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The Johnson homomorphism and the third rational cohomology group of the Torelli group

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Abstract

Using the Johnson homomorphism and the representation theory of the symplectic group, we study the third rational cohomology group of the Torelli group, which gives characteristic classes of surface bundles whose holonomy groups act trivially on the homology groups of their fibers.

MSC: primary 55R40; secondary 32G15, 57R20

Keywords: Mapping class group; Torelli group; Johnson homomorphism

1. Introduction

Let Σ_g be a closed oriented surface of genus $g \geqslant 3$ and let $\mathcal{M}_g$ be its mapping class group, namely it is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. $\mathcal{M}_g$ acts on the first homology group $H := H_1(\Sigma_g)$ of Σ_g and it gives the classical representation

$$\mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z})$$

with the kernel $\mathcal{I}_g$ called the Torelli group, which is the main subject of this paper.

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To study the structure of $\mathcal{I}_g$, Johnson defined a surjective homomorphism

$$\tau : \mathcal{I}_g \to \Big(\bigwedge^3 H\Big)/H$$

in [6] where H is considered to be a subgroup of $\bigwedge^3 H$ as mentioned in Section 3. This homomorphism is now called the (first) Johnson homomorphism and many properties of $\mathcal{I}_g$ have been derived from this.

In this paper, we use this homomorphism to investigate the rational cohomology group of $\mathcal{I}_g$. In particular, we consider the induced homomorphism

$$\tau^* : H^n\Big(\Big(\bigwedge^3 H\Big)/H, \mathbb{Q}\Big) \to H^n(\mathcal{I}_g, \mathbb{Q}).$$

The case of $n = 1$ was studied by Johnson in [7] where he showed that τ^* is an isomorphism. The case of $n = 2$ was settled by Hain in [5] where he showed that there exist non-trivial elements of $\operatorname{Ker}\tau^*$ and described them in terms of the representation theory of the symplectic group. Now we treat the case of $n = 3$ by using the method similar to that of Hain. Main results are described in Section 5. To each irreducible component, except one, of $H^3((\bigwedge^3 H)/H, \mathbb{Q}) \cong \bigwedge^3((\bigwedge^3 H)/H) \otimes \mathbb{Q}$ with respect to the action of $\mathrm{Sp}(2g, \mathbb{Q})$, we determine whether it survives in $H^3(\mathcal{I}_g, \mathbb{Q})$ or not. Then we show that there exists a relationship between the non-triviality of the last one and that of certain characteristic classes of surface bundles whose holonomy groups are contained in the Torelli group.

2. Homology and cohomology of groups

In this paper, we mainly use the homology and cohomology of groups. We refer to Brown's book [2] for the general theory of them. Here we briefly review their definitions and some properties.

Let G be a group and A be $\mathbb{Z}$ or $\mathbb{Q}$. We regard A as a G-module with the trivial G-action. Then the homology and cohomology of G with coefficients in A are defined by

$$H_*(G, A) := H_*\big(K(G,1), A\big),$$
$$H^*(G, A) := H^*\big(K(G,1), A\big),$$

where H_* and H^* in the right-hand side are the ordinary homology and cohomology, and $K(G,1)$ is the Eilenberg–MacLane space of type $(G,1)$. By definition, we have

$$H_0(G, \mathbb{Z}) \cong H^0(G, \mathbb{Z}) \cong \mathbb{Z},$$
$$H_1(G, \mathbb{Z}) \cong G/[G,G].$$

Moreover, we have

$$H_*(G, \mathbb{Q}) \cong H_*(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$H^*(G, \mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Z}}\big(H_*(G,\mathbb{Z}), \mathbb{Q}\big) \cong \mathrm{Hom}_{\mathbb{Q}}\big(H_*(G,\mathbb{Q}), \mathbb{Q}\big)$$

in general.

For later use, we observe the homology and cohomology of the finitely generated free Abelian group $\mathbb{Z}^n$ further. Since $K(\mathbb{Z}^n, 1) = T^n$, we have

$$H^*(\mathbb{Z}^n, \mathbb{Z}) \cong H^*(T^n, \mathbb{Z}) \cong \bigwedge^* H^1(T^n, \mathbb{Z}) \cong \bigwedge^* \mathbb{Z}^n$$

where the wedge product in $\bigwedge^* \mathbb{Z}^n$ corresponds to the cup product in $H^*(\mathbb{Z}^n, \mathbb{Z})$. As for the homology, we have

$$H_*(\mathbb{Z}^n, \mathbb{Z}) \cong H_*(T^n, \mathbb{Z}) \cong \bigwedge^* H_1(T^n, \mathbb{Z}) \cong \bigwedge^* \mathbb{Z}^n$$

where the wedge product in $\bigwedge^* \mathbb{Z}^n$ corresponds to the Pontryagin product in $H_*(\mathbb{Z}^n, \mathbb{Z})$. The Pontryagin product is defined for the homology of general Abelian groups and, in this case, it is the composite of

$$H_k(\mathbb{Z}^n, \mathbb{Z}) \otimes H_l(\mathbb{Z}^n, \mathbb{Z}) \xrightarrow{\times} H_{k+l}(\mathbb{Z}^n \times \mathbb{Z}^n, \mathbb{Z} \otimes \mathbb{Z}) \longrightarrow H_{k+l}(\mathbb{Z}^n, \mathbb{Z})$$

where the first map is the cross product and the second one is induced by the multiplication map $((g, g') \mapsto g + g', \lambda \otimes \lambda' \mapsto \lambda\lambda')$ for $g, g' \in \mathbb{Z}^n$ and $\lambda, \lambda' \in \mathbb{Z}$. The Pontryagin product makes it easier to treat the homology of finitely generated free Abelian groups in higher degrees. Let $x_1, x_2, \ldots, x_n$ denote the standard basis elements of $\mathbb{Z}^n$. Then the homology class $1 \in \mathbb{Z} \cong H_n(\mathbb{Z}^n, \mathbb{Z})$ which corresponds to the fundamental class of the torus T^n is given by

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n \in \bigwedge^n \mathbb{Z}^n \cong H_n(\mathbb{Z}^n, \mathbb{Z}).$$

We again call it the fundamental class. The following lemma is often used in later sections.

Lemma 2.1. *Let A be a finitely generated free Abelian group and $f : \mathbb{Z}^n \to A$ be a group homomorphism. Then the image of the fundamental class $1 \in \mathbb{Z} \cong H_n(\mathbb{Z}^n, \mathbb{Z})$ by f_* is*

$$f(x_1) \wedge f(x_2) \wedge \cdots \wedge f(x_n) \in \bigwedge^n A \cong H_n(A).$$

Proof. The induced homomorphism $f_* : H_1(\mathbb{Z}^n, \mathbb{Z}) \to H_1(A, \mathbb{Z})$ is f itself. Then the lemma follows from the fact that f_* preserves the Pontryagin product. □

3. Background

Let Σ_g be a closed oriented surface of genus $g \geqslant 3$ and $H_1(\Sigma_g)$ be its first homology group with coefficients in $\mathbb{Z}$. We fix a symplectic basis $\langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle$ of $H_1(\Sigma_g)$ as in Fig. 1. The Poincaré duality supplies a canonical isomorphism of $H_1(\Sigma_g)$ with its dual module $\mathrm{Hom}(H_1(\Sigma_g), \mathbb{Z}) = H^1(\Sigma_g)$, the first cohomology group of Σ_g with coefficients in $\mathbb{Z}$. In this isomorphism, a_i (respectively b_i) $\in H_1(\Sigma_g)$ corresponds to $-b_i^*$ (respectively a_i^*) $\in H^1(\Sigma_g)$ where $\langle a_1^*, \ldots, a_g^*, b_1^*, \ldots, b_g^* \rangle$ is the dual basis of $H^1(\Sigma_g)$. We use the same symbol H for these canonically isomorphic Abelian groups.

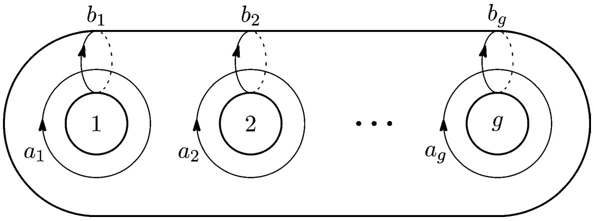


Fig. 1.

Let $\mathrm{Diff}_+\Sigma_g$ denote the topological group of all orientation preserving diffeomorphisms of Σ_g with the C^∞ topology. By a result of Earle and Eells [3], the identity component of $\mathrm{Diff}_+\Sigma_g$ is contractible so that $\mathrm{Diff}_+\Sigma_g$ is homotopy equivalent to the *mapping class group* $\mathcal{M}_g$ *of* Σ_g, which is the group of path components of $\mathrm{Diff}_+\Sigma_g$. Hence the classifying space $\mathrm{BDiff}_+\Sigma_g$ is homotopy equivalent to $\mathrm{B}\mathcal{M}_g = K(\mathcal{M}_g, 1)$. This means that cohomology classes of $\mathcal{M}_g$ give characteristic classes of oriented Σ_g-bundles and enables us to treat the theory of oriented Σ_g-bundles from the algebraic point of view.

Our strategy to understand the structure of $\mathcal{M}_g$ is to approximate it by an object which is easier to access. It is done by making homomorphisms from or to $\mathcal{M}_g$. The fundamental one is given by the action of $\mathcal{M}_g$ on H. This action yields the classical representation

$$\mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z})$$

which is known to be surjective. The kernel of this homomorphism is called the *Torelli group* and denoted by $\mathcal{I}_g$. From an argument similar to the above, cohomology classes of $\mathcal{I}_g$ give characteristic classes of oriented Σ_g-bundles whose holonomy groups act trivially on homology groups of their fibers.

In [6], Johnson defined a surjective homomorphism

$$\tau : \mathcal{I}_g \to \Bigl(\bigwedge^3 H\Bigr)/H$$

which is now called the (*first*) *Johnson homomorphism*. Here H is considered as a subgroup of $\bigwedge^3 H$ by the injection

$$H \hookrightarrow \bigwedge^3 H \quad \left(x \mapsto x \wedge \left(\sum_{i=1}^{g} a_i \wedge b_i\right)\right)$$

which is $\mathrm{Sp}(2g, \mathbb{Z})$-equivariant. From now on we denote $(\bigwedge^3 H)/H$ by U, for short. It is easy to see that U is a free Abelian group of rank $\binom{2g}{3} - 2g$. Before reviewing the definition of τ, we introduce the *pointed mapping class group* $\mathcal{M}_{g,*}$. It is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g relative to the base point $* \in \Sigma_g$. We denote the corresponding Torelli group by $\mathcal{I}_{g,*}$. Then we have the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{I}_{g,*} & \longrightarrow & \mathcal{M}_{g,*} & \longrightarrow & \mathrm{Sp}(2g, \mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathcal{I}_g & \longrightarrow & \mathcal{M}_g & \longrightarrow & \mathrm{Sp}(2g, \mathbb{Z}) & \longrightarrow & 1
\end{array}$$

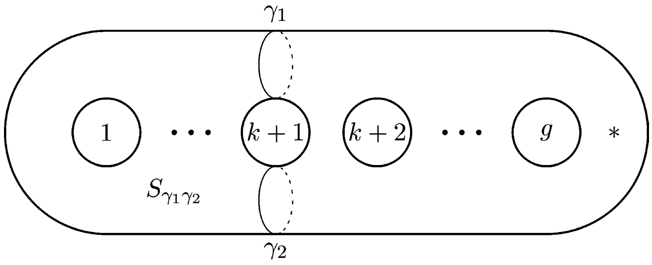


Fig. 2.

where vertical arrows are surjective homomorphisms induced by forgetting the base point.

In the pointed case, the Johnson homomorphism has $\bigwedge^3 H$ as its target group and is given as follows. First, note that $\mathcal{I}_{g,*}$ is generated by all *BP maps*. A *BP map of genus k* has the form of $T_{\gamma_1}T_{\gamma_2}^{-1}$ where T_{γ_i} denotes the Dehn twist along the simple closed curve γ_i on Σ_g given as in Fig. 2. We define the genus of a BP map $T_{\gamma_1}T_{\gamma_2}^{-1}$ to be the genus of the subsurface $S_{\gamma_1\gamma_2}$ of Σ_g bounded by γ_1, γ_2 and *not* containing the base point $*$.

For a given BP map $T_{\gamma_1}T_{\gamma_2}^{-1}$ of genus k with the subsurface $S_{\gamma_1\gamma_2}$, choose a maximal symplectic subspace V of $H_1(S_{\gamma_1\gamma_2})$ with a symplectic basis x_i, y_i $(i=1,\ldots,k)$. Then $H_1(S_{\gamma_1\gamma_2})$ is a free Abelian group generated by x_i, y_i $(i=1,\ldots,k)$ and c where c is the homology class of γ_1 given by the orientation putting $S_{\gamma_1\gamma_2}$ on its left as we move around γ_1. The value of τ on $T_{\gamma_1}T_{\gamma_2}^{-1}$ by τ is defined to be

$$\tau\big(T_{\gamma_1}T_{\gamma_2}^{-1}\big)=\left(\sum_{i=1}^{n} a_i\wedge b_i\right)\wedge c$$

which does not depend on the choice of the symplectic basis. This correspondence determines a well-defined homomorphism $\tau:\mathcal{I}_{g,*}\to\bigwedge^3 H$. A direct computation shows that the image of $\mathrm{Ker}(\mathcal{I}_{g,*}\to\mathcal{I}_g)$ is contained in $H\subset\bigwedge^3 H$ so that we can define the homomorphism $\tau:\mathcal{I}_g\to U$ to be the one which makes the following diagram:

$$\begin{tikzcd}
\mathcal{I}_{g,*} \arrow[r, "\tau"] \arrow[d] & \bigwedge^3 H \arrow[d] \\
\mathcal{I}_g \arrow[r, "\tau"] & U
\end{tikzcd}$$

commutative where the right vertical map is the natural projection. That is, $\tau:\mathcal{I}_g\to U$ is given by

$$\tau(\varphi)=\tau(\tilde{\varphi}) \bmod H$$

for $\varphi\in\mathcal{I}_g$ where $\tilde{\varphi}\in\mathcal{I}_{g,*}$ is any lift of φ. We use the same symbol τ for the above two versions of Johnson homomorphisms but any confusion will not occur.

We mention two important properties of the Johnson homomorphism $\tau:\mathcal{I}_g\to U$ which we effectively use later.

Theorem 3.1 (Johnson [6,7]).

(1) *τ is $\mathcal{M}_g$-equivariant where $\mathcal{M}_g$ acts on $\mathcal{I}_g$ by the outer conjugation and acts on U by the way induced from the action of $\mathcal{M}_g$ on H.*
(2) *The induced homomorphism $\tau_* : H_1(\mathcal{I}_g, \mathbb{Z}) \to H_1(U, \mathbb{Z}) = U$ is an isomorphism modulo 2-torsion.*

From this theorem, we see that τ gives a good approximation of the abelianization of $\mathcal{I}_g$ by a free Abelian group U. In the pointed case, there exist properties similar to the above.

The Johnson homomorphism is extended from $\mathcal{I}_g$ to the whole of $\mathcal{M}_g$ by Morita in [13] and has the following form:

$$\begin{array}{ccc} \mathcal{M}_{g,*} & \xrightarrow{\rho_1} & \frac{1}{2}\bigwedge^3 H \rtimes \mathrm{Sp}(2g,\mathbb{Z}) \\ \cup & & \cup \\ \mathcal{I}_{g,*} & \xrightarrow{\tau} & \bigwedge^3 H \end{array} \qquad \begin{array}{ccc} \mathcal{M}_{g} & \xrightarrow{\rho_1} & \frac{1}{2} U \rtimes \mathrm{Sp}(2g,\mathbb{Z}) \\ \cup & & \cup \\ \mathcal{I}_{g} & \xrightarrow{\tau} & U \end{array}$$

where $\frac{1}{2}\bigwedge^3 H$ is a submodule of $\bigwedge^3(H \otimes_{\mathbb{Z}} \mathbb{Q})$ and $\frac{1}{2}U$ is similar. ρ_1 is called the *extended Johnson homomorphism*. When we consider ρ_1 to be an approximation of $\mathcal{M}_g$ by $\frac{1}{2}U \rtimes \mathrm{Sp}(2g,\mathbb{Z})$, our next task is to measure the gap between them. We will see it from the view of the rational cohomology, namely, we consider induced homomorphisms

$$\rho_1^* : H^*\left(\frac{1}{2}\bigwedge^3 H \rtimes \mathrm{Sp}(2g,\mathbb{Z}), \mathbb{Q}\right) \to H^*(\mathcal{M}_{g,*}, \mathbb{Q}),$$
$$\rho_1^* : H^*\left(\frac{1}{2} U \rtimes \mathrm{Sp}(2g,\mathbb{Z}), \mathbb{Q}\right) \to H^*(\mathcal{M}_{g}, \mathbb{Q}).$$

In [9], Kawazumi and Morita determined $\mathrm{Image}\,\rho_1^*$ completely. To mention their results, we recall the definitions of the Euler class and Morita–Mumford classes which are characteristic classes of oriented Σ_g-bundles defined independently in [11,18]. We now follow the conventions of [11]. Let $\pi : \mathrm{EDiff}_+ \Sigma_g \to \mathrm{BDiff}_+ \Sigma_g$ be the universal oriented Σ_g-bundle. Then the Morita–Mumford class $e_i \in H^{2i}(\mathrm{BDiff}_+ \Sigma_g, \mathbb{Z})$ is defined by $e_i = \pi_*(e^{i+1})$ where $e \in H^2(\mathrm{EDiff}_+ \Sigma_g, \mathbb{Z})$ is the Euler class of the relative tangent bundle of π and π_* is the Gysin map. Since $H^*(\mathrm{EDiff}_+ \Sigma_g, \mathbb{Z}) = H^*(\mathcal{M}_{g,*}, \mathbb{Z})$ and $H^*(\mathrm{BDiff}_+ \Sigma_g, \mathbb{Z}) = H^*(\mathcal{M}_g, \mathbb{Z})$, we can consider that $e \in H^2(\mathcal{M}_{g,*}, \mathbb{Z})$ and $e_i \in H^{2i}(\mathcal{M}_g, \mathbb{Z})$. We write $e_i \in H^{2i}(\mathcal{M}_{g,*}, \mathbb{Z})$ again for the pull-back of $e_i \in H^{2i}(\mathcal{M}_g, \mathbb{Z})$ under the homomorphism $\mathcal{M}_{g,*} \to \mathcal{M}_g$. Using them, we mention the following theorem:

Theorem 3.2 (Kawazumi, Morita [9]).

$$\mathrm{Image}\,\rho_1^* = \mathbb{Q}[e, e_1, e_2, \ldots]/(\text{relations}) \subset H^*(\mathcal{M}_{g,*}, \mathbb{Q}),$$
$$\mathrm{Image}\,\rho_1^* = \mathbb{Q}[e_1, e_2, \ldots]/(\text{relations}) \subset H^*(\mathcal{M}_{g}, \mathbb{Q}).$$

For the proof of this theorem, we refer to [9]. This theorem shows that the extended Johnson homomorphism ρ_1 captures an important part of $H^*(\mathcal{M}_{g,*}, \mathbb{Q})$ or $H^*(\mathcal{M}_g, \mathbb{Q})$. Then the following problem naturally occurs to us.

Problem. Let $\tau : \mathcal{I}_g \to U$ be the Johnson homomorphism. Determine the image and the kernel of the induced homomorphism

$$\tau^* : H^*(U, \mathbb{Q}) \to H^*(\mathcal{I}_g, \mathbb{Q}).$$

In later sections, we treat this problem in cases of degrees up to 3, starting from results of Johnson and Hain. Note that this problem contains that of the non-triviality of powers of the Euler class e^k $(k \geqslant 2)$ and even Morita–Mumford classes e_{2i} $(i \geqslant 1)$ on the Torelli group. For the present, however, we have little information about it. As for odd Morita–Mumford classes e_{2i-1} $(i \geqslant 1)$, it is known that they are trivial in $H^*(\mathcal{I}_g, \mathbb{Q})$. See [11,15] for details.

4. The cohomology of the Torelli group and the representation theory of the symplectic group

In this section, we begin our study of the problem mentioned in the previous section, namely we consider the homomorphism

$$\tau^* : H^*(U, \mathbb{Q}) \to H^*(\mathcal{I}_g, \mathbb{Q})$$

and its kernel. The case of $* = 1$ is settled by using Theorem 3.1(2) due to Johnson that $\tau^* : H^1(U, \mathbb{Q}) \to H^1(\mathcal{I}_g, \mathbb{Q})$ is an isomorphism.

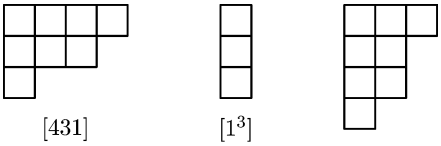
Before we proceed further, we give some remarks about our setting of the homomorphism τ^*. As mentioned in Section 2, the source $H^*(U, \mathbb{Q})$ of τ^* is isomorphic to $\bigwedge^* U_{\mathbb{Q}}$ where we write $U_{\mathbb{Q}}$ for $U \otimes_{\mathbb{Z}} \mathbb{Q}$. The rational dimension of this vector space grows very rapidly as $*$ and g increase. Hence we need some devices to treat it efficiently. We now adopt the representation theory of the symplectic group $\mathrm{Sp}(2g, \mathbb{Q})$ for the following reason.

The action of $\mathrm{Sp}(2g, \mathbb{Z})$ on H naturally extends to an action of $\mathrm{Sp}(2g, \mathbb{Q})$ on $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$. Then we can consider $\bigwedge^* U_{\mathbb{Q}}$ to be an $\mathrm{Sp}(2g, \mathbb{Q})$-vector space. On the other hand, since $\tau : \mathcal{I}_g \to U$ is $\mathcal{M}_g$-equivariant, $\tau^* : \bigwedge^n U_{\mathbb{Q}} \to H^n(\mathcal{I}_g, \mathbb{Q})$ is also $\mathcal{M}_g$-equivariant. In particular, $\operatorname{Ker} \tau^*$ becomes an $\mathrm{Sp}(2g, \mathbb{Z})$-subspace. Moreover, we can say the following:

Lemma 4.1. $\operatorname{Ker} \tau^*$ *is an* $\mathrm{Sp}(2g, \mathbb{Q})$*-subspace.*

Proof. For the proof of this lemma, we refer to the proof of Lemma 2.2.8 in [1], where more general situations are treated. □

From this lemma, our problem becomes as follows: Determine the $\mathrm{Sp}(2g, \mathbb{Q})$-subspace $\operatorname{Ker} \tau^*$ in the $\mathrm{Sp}(2g, \mathbb{Q})$-vector space $\bigwedge^* U_{\mathbb{Q}}$. For such a situation, the representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$ plays an essential role. That is, for a given non-zero vector v which is in some irreducible component of $\bigwedge^* U_{\mathbb{Q}}$ with respect to the action of $\mathrm{Sp}(2g, \mathbb{Q})$, if v is in $\operatorname{Ker} \tau^*$, then all vectors in this irreducible component are in $\operatorname{Ker} \tau^*$ because of the irreducibility.



$[431]$ $[1^3]$ $[32^21]$

Fig. 3.

Here we summarize the notation and general facts concerning the representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$ from [4,5,15,16]. First we consider the Lie group $\mathrm{Sp}(2g, \mathbb{C})$ and its Lie algebra $\mathfrak{sp}(2g, \mathbb{C})$. By the general theory of the representation, we see that finite dimensional representations of $\mathrm{Sp}(2g, \mathbb{C})$ coincide with those of $\mathfrak{sp}(2g, \mathbb{C})$. Their common irreducible representations (up to isomorphisms) are parameterized by Young diagrams whose numbers of rows are less than or equal to g. We call such young diagrams admissible. These representations are all rational representations defined over $\mathbb{Q}$ so that we can consider them as irreducible representations of $\mathrm{Sp}(2g, \mathbb{Q})$ and $\mathfrak{sp}(2g, \mathbb{Q})$. We follow the notation in [15] to describe Young diagrams as in Fig. 3.

The correspondence between admissible Young diagrams and irreducible representations are explicitly given as follows. First we assign $\mathbb{Q}$, which is the trivial irreducible representation, to the Young diagram [0] and assign $H_\mathbb{Q}$, which is the fundamental representation of $\mathfrak{sp}(2g, \mathbb{Q})$, to the Young diagram [1]. We fix a symplectic basis $\langle a_1, \ldots, a_g, b_1, \ldots, b_g\rangle$ of $H_\mathbb{Q} = [1]$ with respect to the intersection form $\mu : H_\mathbb{Q} \otimes H_\mathbb{Q} \to \mathbb{Q}$, which is a non-degenerate skew symmetric bilinear form, namely a_i, b_i $(i = 1, \ldots, g)$ satisfy the conditions

$$\mu(a_i, a_j) = 0, \qquad \mu(b_i, b_j) = 0, \qquad \mu(a_i, b_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker's delta. In the general case, the Young diagram $[n_1 n_2 \cdots n_l]$, where n_i are integers satisfying $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_l \geqslant 1$ and $l \leqslant g$, corresponds to the $\mathfrak{sp}(2g, \mathbb{Q})$-vector space V given as follows. Let $[m_1 m_2 \cdots m_k]$ be the Young diagram obtained by transposing $[n_1 n_2 \cdots n_l]$. Then V is defined to be the irreducible $\mathfrak{sp}(2g, \mathbb{Q})$-subspace of

$$\Bigl(\bigwedge^{m_1} H_\mathbb{Q}\Bigr) \otimes \Bigl(\bigwedge^{m_2} H_\mathbb{Q}\Bigr) \otimes \cdots \otimes \Bigl(\bigwedge^{m_k} H_\mathbb{Q}\Bigr)$$

containing the vector

$$(a_1 \wedge a_2 \wedge \cdots \wedge a_{m_1}) \otimes (a_1 \wedge a_2 \wedge \cdots \wedge a_{m_2}) \otimes \cdots \otimes (a_1 \wedge a_2 \wedge \cdots \wedge a_{m_k})$$

which is called the *highest weight vector of* $[n_1 n_2 \cdots n_l]$.

As an example, we consider $U_\mathbb{Q}$ and $\bigwedge^3 H_\mathbb{Q}$ in the present contexts. Let $C_3 : \bigwedge^3 H_\mathbb{Q} \to H_\mathbb{Q}$ be the contraction given by

$$C_3(x \wedge y \wedge z) = \mu(x, y)z + \mu(y, z)x + \mu(z, x)y$$

and a natural injection $H_\mathbb{Q} \hookrightarrow \bigwedge^3 H_\mathbb{Q}$ given by

$$x \mapsto x \wedge \omega$$

where $\omega := \sum_{i=1}^{g} a_i \wedge b_i \in \bigwedge^2 H_{\mathbb{Q}}$ is the symplectic class. Then we can identify $\operatorname{Ker} C_3$ with $U_{\mathbb{Q}} = (\bigwedge^3 H_{\mathbb{Q}})/H_{\mathbb{Q}}$. In particular, $U_{\mathbb{Q}}$ is isomorphic to the $\mathfrak{sp}(2g, \mathbb{Q})$-subspace of $\bigwedge^3 H_{\mathbb{Q}}$ containing $a_1 \wedge a_2 \wedge a_3$ which is irreducible. Hence $U_{\mathbb{Q}}$ is the irreducible representation which corresponds to the Young diagram $[1^3]$ and the irreducible decomposition of $\bigwedge^3 H_{\mathbb{Q}}$ is given by

$$\bigwedge^3 H_{\mathbb{Q}} = H_{\mathbb{Q}} \oplus U_{\mathbb{Q}} = [1] + \left[1^3\right]$$

where we write $+$ for the operation of direct sum. The above decomposition is explicitly given as follows. Let $q : \bigwedge^3 H_{\mathbb{Q}} \to \bigwedge^3 H_{\mathbb{Q}}$ be a homomorphism defined by

$$q(\xi) = \xi - \frac{1}{g-1} C_3(\xi) \wedge \omega \quad \left(\xi \in \bigwedge^3 H_{\mathbb{Q}}\right)$$

for $\xi \in \bigwedge^3 H_{\mathbb{Q}}$. Then we can see that $\operatorname{Image} q = \operatorname{Ker} C_3 = U_{\mathbb{Q}}$ and the direct sum decomposition of $\bigwedge^3 H_{\mathbb{Q}} = U_{\mathbb{Q}} \oplus H_{\mathbb{Q}}$ is given by the correspondence $\xi \mapsto (q(\xi), \frac{1}{g-1} C_3(\xi) \wedge \omega)$.

Now we define

$$\left.\begin{aligned} p_{ij} &= q(a_i \wedge a_j \wedge b_j) = a_i \wedge a_j \wedge b_j - \frac{1}{g-1} a_i \wedge \omega \\ q_{ij} &= q(b_i \wedge a_j \wedge b_j) = b_i \wedge a_j \wedge b_j - \frac{1}{g-1} b_i \wedge \omega \end{aligned}\right. \quad (i \neq j).$$

It is easily checked that there are $2g$ relations

$$\sum_{j \neq i} p_{ij} = 0, \quad \sum_{j \neq i} q_{ij} = 0 \quad (i = 1, \ldots, g).$$

Then we obtain an explicit description of $U_{\mathbb{Q}}$ as follows:

Lemma 4.2 (Morita [16]). *$U_{\mathbb{Q}}$ is, as a $\mathbb{Q}$-vector space, spanned by the following elements*

$$\begin{aligned} &a_i \wedge a_j \wedge a_k, \quad b_i \wedge b_j \wedge b_k \quad (i < j < k), \\ &a_i \wedge a_j \wedge b_k, \quad b_i \wedge b_j \wedge a_k \quad (i < j,\ k \neq i, j), \\ &p_{ij}, \quad q_{ij} \quad (i \neq j) \end{aligned}$$

and $2g$ relations of $\sum_{j \neq i} p_{ij} = 0$, $\sum_{j \neq i} q_{ij} = 0$ $(i = 1, 2, \ldots, g)$ represent a complete system of linear relations among them.

Note that U can be considered as a lattice in $U_{\mathbb{Q}}$. That is, U is isomorphic to the $\mathrm{Sp}(2g, \mathbb{Z})$-submodule of $U_{\mathbb{Q}}$ generated by above generators.

For later use, we define the following elements $X_{i,j}, Y_{i,j}$ $(i \neq j)$ and U_i of $\mathfrak{sp}(2g, \mathbb{Q})$ characterized by their actions on $H_{\mathbb{Q}}$ as follows:

$$\begin{aligned} &X_{i,j}(a_k) = \delta_{jk} a_i, && X_{i,j}(b_k) = -\delta_{ik} b_j, \\ &Y_{i,j}(a_k) = 0, && Y_{i,j}(b_k) = \delta_{ik} a_j + \delta_{jk} a_i, \\ &U_i(a_k) = 0, && U_i(b_k) = \delta_{ik} a_i. \end{aligned}$$

We can check that $X_{i,j}, Y_{i,j}, U_i$ are certainly elements of $\mathfrak{sp}(2g,\mathbb{Q})$.

We also make use of following four types of $\mathfrak{sp}(2g,\mathbb{Q})$-equivariant homomorphisms frequently. Recall that $\mu : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$ is the intersection form on $H_{\mathbb{Q}}$.

(1) The *contraction* $C_k : \bigwedge^k H_{\mathbb{Q}} \to \bigwedge^{k-2} H_{\mathbb{Q}}$ $(k = 2, 3, \ldots)$ is given by

$$C_k(x_1 \wedge \cdots \wedge x_k) = \sum_{1 \leqslant i < j \leqslant k} (-1)^{i+j+1} \mu(x_i, x_j) x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_k$$

where $\widehat{x_i}$ means to exclude x_i and we define $\bigwedge^0 H_{\mathbb{Q}}$ to be the trivial representation $\mathbb{Q}$. It is known that the kernel of C_k is the irreducible $\mathfrak{sp}(2g,\mathbb{Q})$-vector space corresponding to the Young diagram $[1^k]$. (See [4, Theorem 17.5].)

Let V be some $\mathfrak{sp}(2g,\mathbb{Q})$-vector space and let v_i be elements of V.

(2) The *canonical inclusion* $i_V^n : \bigwedge^n V \hookrightarrow \otimes^n V$ is given by

$$i_V^n(v_1 \wedge v_2 \wedge \cdots \wedge v_n) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$

where $\mathfrak{S}_n$ is the symmetric group of degree n.

(3) The *multiplication* $\varphi_V^{m,n} : (\bigwedge^m V) \otimes (\bigwedge^n V) \to \bigwedge^{m+n} V$ is given by

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge v_{m+2} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_{m+n}.$$

We define the multiplication $\psi_V^{m,n} : \bigwedge^m (\bigwedge^n V) \to \bigwedge^{mn} V$ similarly.

(4) Using (2) and the canonical projection $\bigotimes^2 V \to \bigwedge^2 V$ given by $v_1 \otimes v_2 \mapsto v_1 \wedge v_2$, we also define an inclusion $j_V : \bigwedge^3 V \hookrightarrow V \otimes (\bigwedge^2 V)$ by

$$j_V(v_1 \wedge v_2 \wedge v_3) = v_1 \otimes (v_2 \wedge v_3) + v_2 \otimes (v_3 \wedge v_1) + v_3 \otimes (v_1 \wedge v_2).$$

For simplicity, we denote $a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_n}$ by $A_{i_1 i_2 \cdots i_n}$. As for indices which are negative, we interpret them by changing a into b. For example, A_{1-2-34} stands for $a_1 \wedge b_2 \wedge b_3 \wedge a_4$.

Now we return to our problem of determining $\mathrm{Ker}(\tau^* : \bigwedge^* U_{\mathbb{Q}} \to H^*(\mathcal{I}_g, U_{\mathbb{Q}}))$. The case of $* = 2$ was settled by Hain in [5]. We review his argument in Section 10 of [5] in the present contexts as follows.

First, we see the irreducible decomposition of $\bigwedge^2 U_{\mathbb{Q}}$.

Lemma 4.3 (Hain [5]). *If $g \geqslant 3$, then the irreducible decomposition of $\bigwedge^2 U_{\mathbb{Q}}$ is given by*

$$\bigwedge^2 U_{\mathbb{Q}} = \begin{cases} [0] + [2^2] + [1^2] + [2^2 1^2] + [1^4] + [1^6] & (g \geqslant 6), \\ [0] + [2^2] + [1^2] + [2^2 1^2] + [1^4] & (g = 5), \\ [0] + [2^2] + [1^2] + [2^2 1^2] & (g = 4), \\ [0] + [2^2] & (g = 3). \end{cases}$$

Note that the notation for describing irreducible representations in [5] is different from ours. For example, $V(\lambda_1)$ corresponds to [1] and $V(\lambda_3)$ corresponds to $[1^3]$. As mentioned in Remark 6.2 of [5], the irreducible decomposition stabilizes for sufficiently large g. In this case the stability range is $g \geqslant 6$.

We give some remarks about methods for obtaining the irreducible decomposition of $\mathfrak{sp}(2g,\mathbb{Q})$-vector spaces. In [8,10], methods for decomposing the tensor product of two irreducible representations are given. Using them, for example, we can obtain the irreducible representation of $\bigwedge^2 U_{\mathbb{Q}} = \wedge[1^3]$ with fixed g as follows. Note that there exists a surjective homomorphism $\bigotimes^2 U_{\mathbb{Q}} \to \bigwedge^2 U_{\mathbb{Q}}$ so that the components which appear in the irreducible decomposition of $\bigwedge^2 U_{\mathbb{Q}}$ are also contained in that of $\bigotimes^2 U_{\mathbb{Q}}$. Hence by finding irreducible components of $\bigotimes^2 U_{\mathbb{Q}}$ which are also contained in $\bigwedge^2 U_{\mathbb{Q}}$ and taking the sum of dimensions of irreducible representations, which are calculated by Weyl's dimensional formula, until the total dimension coincides with the dimension of $\bigwedge^2 U_{\mathbb{Q}}$, we obtain the irreducible decomposition of $\bigwedge^2 U_{\mathbb{Q}}$. Another method is to use the computer program LiE. Once we obtain the data of the irreducible decomposition, it is easy to check it by hand by using the same method as above.

With respect to the above irreducible decomposition of $\bigwedge^2 U_{\mathbb{Q}}$, Hain showed the following theorem.

Theorem 4.4 (Hain [5]). *For all* $g \geqslant 3$, $\operatorname{Ker}\tau^* = [0] + [2^2]$.

This statement corresponds to [5, Theorem 10.1]. In [14], Morita explained this theorem by using methods closer to those of this paper. Here we review the method of showing that the other components except [0] and $[2^2]$ are not contained in $\operatorname{Ker}\tau^*$. Notice that an irreducible component is *not* in $\operatorname{Ker}(\tau^* : \bigwedge^2 U_{\mathbb{Q}} \to H^2(\mathcal{I}_g,\mathbb{Q}))$ if and only if it is contained in $\operatorname{Image}(\tau_* : H_2(\mathcal{I}_g,\mathbb{Q}) \to H_2(U,\mathbb{Q}) \cong \bigwedge^2 U_{\mathbb{Q}})$. To obtain an element in $H_2(\mathcal{I}_g,\mathbb{Q})$, Hain used a fundamental class of an Abelian subgroup of $\mathcal{I}_g$, so that he obtained an element of $H_2(\mathcal{I}_g,\mathbb{Q})$ whose image under τ_* is given by

$$v = \big((g-1)a_1 \wedge a_2 \wedge b_2 - a_1 \wedge \omega\big) \wedge \big((g-1)a_3 \wedge a_4 \wedge b_4 - a_3 \wedge \omega\big).$$

Lemma 2.1 is used to do the calculation involved here. Then he showed the result by decomposing v to each irreducible component. For example, v goes to a non-trivial element in $[1^6] \subset \bigwedge^6 H_{\mathbb{Q}}$ by the map which is the composition of

$$\bigwedge^2 U_{\mathbb{Q}} \hookrightarrow \bigwedge^2\Big(\bigwedge^3 H_{\mathbb{Q}}\Big) \xrightarrow{\psi^{2,3}_{H_{\mathbb{Q}}}} \bigwedge^6 H_{\mathbb{Q}}.$$

This shows that $[1^6]$ are contained in $\operatorname{Image}(\tau_* : H_2(\mathcal{I}_g,\mathbb{Q}) \to H_2(U,\mathbb{Q}) \cong \bigwedge^2 U_{\mathbb{Q}})$.

As an immediate corollary of this theorem, we see that when $g = 3$, τ^* is trivial for all degrees greater than 0.

5. Main results

In the rest of this paper, we treat the case of $* = 3$. First, we need to know the irreducible decomposition of $\bigwedge^3 U_{\mathbb{Q}}$. In this case the stability range is given by $g \geqslant 9$.

Table 1

	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$	$g=8$	$g\geqslant 9$
$[3^21^3]$			1	1	1	1	1
$[3^21]$		1	1	1	1	1	1
$[32^3]$		1	1	1	1	1	1
$[321^2]$		1	1	1	1	1	1
$[32]$	1	1	1	1	1	1	1
$[2^31^3]$				1	1	1	1
$[2^31]$			1	1	1	1	1
$[2^21^5]$					1	1	1
$[2^21^3]$			1	2	2	2	2
$[2^21]$		1	2	2	2	2	2
$[21^5]$				1	1	1	1
$[21^3]$		1	2	2	2	2	2
$[21]$		1	1	1	1	1	1
$[1^9]$							1
$[1^7]$						1	1
$[1^5]$			1	1	2	2	2
$[1^3]$	1	2	2	3	3	3	3
$[1]$			1	1	1	1	1

Lemma 5.1. *The irreducible decomposition of* $\bigwedge^3 U_{\mathbb{Q}}$ *is given by Table* 1, *where numbers indicate multiplicities of the corresponding irredusible representation inside it for each* g.

From the table, we see, for example, that

$$\bigwedge^3 U_{\mathbb{Q}} = \left[3^21\right] + \left[32^3\right] + \left[321^2\right] + [32] + \left[2^21\right] + \left[21^3\right] + [21] + 2\left[1^3\right]$$

when $g=4$.

Proof. Once we obtain the data of the irreducible representation, it is a routine matter to check it by hand as mentioned in the previous section. □

Now we mention the main results of this paper.

Theorem 5.2. *For all* $g\geqslant 9$, $\operatorname{Ker}\tau^*$ *contains the direct sum*

$$\left[3^21\right] + \left[321^2\right] + [32] + \left[2^21^3\right] + \left[2^21\right] + \left[21^3\right] + [21] + 2\left[1^3\right]$$

which is equal to $\operatorname{Image}(\cup : U_{\mathbb{Q}} \otimes ([2^2]+[0]) \to \bigwedge^3 U_{\mathbb{Q}})$. *Moreover, one of the following two possibilities holds*:

(a) $\operatorname{Ker}\tau^* = [3^21] + [321^2] + [32] + [2^21^3] + [2^21] + [21^3] + [21] + 2[1^3]$.
(b) $\operatorname{Ker}\tau^* = ([3^21] + [321^2] + [32] + [2^21^3] + [2^21] + [21^3] + [21] + 2[1^3]) + [1]$.

In Section 6, we also treat the cases of lower genera but here we omit the details. We prove it in Sections 6.1 and 6.2. As for the summand [1], we can relate it with the Euler class and the Morita–Mumford class as follows.

Theorem 5.3. *For all $g \geqslant 5$,*

$$\tau^*\big([1]\big) = \{0\} \subset H^3(\mathcal{I}_g, \mathbb{Q}) \iff e_2 - (2-2g)e^2 = 0 \in H^4(\mathcal{I}_{g,*}, \mathbb{Q})$$

where e is the Euler class and e_2 is the second Morita–Mumford class.

We prove this theorem in Section 6.3. The above condition is compatible with the pull-back of the universal Σ_g-bundle. Therefore comparing the result of Morita in [12] that the pull-back of $e^2 \in H^4(\mathcal{M}_g, \mathbb{Z})$ on the amenable group vanishes, we obtain the next corollary.

Corollary 5.4. *For every amenable group G and every group homomorphism $f: G \to \mathcal{I}_g$,*

$$f^*\tau^*\big([1]\big) = \{0\} \subset H^3(G, \mathbb{Q}).$$

In the proof of Theorem 5.2, we construct some homology classes of $H_3(\mathcal{I}_g, \mathbb{Q})$ which come from fundamental classes of some Abelian subgroups of $\mathcal{I}_g$ to evaluate summands which survive in $H^3(\mathcal{I}_g, \mathbb{Q})$. This corollary implies that on any Abelian group the summand $\tau^*([1])$ is equal to 0 since it is known that

$$(\text{Abelian groups}) \subset (\text{nilpotent groups}) \subset (\text{solvable groups}) \subset (\text{amenable groups}).$$

6. Proofs of the main results

Now we prove Theorems 5.2 and 5.3. In Sections 6.1 and 6.2, all vector spaces are $\mathfrak{sp}(2g, \mathbb{Q})$-vector spaces and all homomorphisms are $\mathfrak{sp}(2g, \mathbb{Q})$-equivariant so that we omit the symbol "$\mathfrak{sp}(2g, \mathbb{Q})$-" for simplicity.

6.1. Summands which are in the kernel

Due to Hain's results, we can obtain some summands in $\operatorname{Ker}\tau^*$ by considering the cup product. That is, we calculate the image of the homomorphism

$$\cup : H^1(U; \mathbb{Q}) \otimes \operatorname{Ker}\big(H^2(U; \mathbb{Q}) \xrightarrow{\tau^*} H^2(\mathcal{I}_g; \mathbb{Q})\big) \to H^3(U; \mathbb{Q}).$$

In terms of $\mathfrak{sp}(2g, \mathbb{Q})$-vector spaces, we determine the image of

$$\wedge : \big[1^3\big] \otimes \big(\big[2^2\big] + [0]\big) \to \bigwedge^3 \big[1^3\big]$$

where $[2^2]+[0]$ is in $\bigwedge^2[1^3]$ and the homomorphism $\wedge$ is given by taking the wedge product.

Lemma 6.1. *If $g \geqslant 3$, the irreducible decomposition of* $[1^3]\otimes([2^2]+[0])$ *is given by*

$$\begin{cases} 2[1^3]+[32]+[21]+[3^21]+[31^2]+[21^3]+[2^21]+[321^2]+[2^21^3] \\ \quad (g\geqslant 5), \\ 2[1^3]+[32]+[21]+[3^21]+[31^2]+[21^3]+[2^21]+[321^2] \\ \quad (g=4), \\ 2[1^3]+[32]+[21]+[3^21]+[31^2] \\ \quad (g=3). \end{cases}$$

Proof. We can obtain the above result by applying the method in [8,10], or by using LiE (with checking the result). Here we omit the detail of the proof. □

With respect to the above decomposition, we prove the following proposition (this fact for the stable range is mentioned in [15, Proposition 6.3] without proof).

Proposition 6.2. *For $g \geqslant 3$, the irreducible decomposition of the image of the homomorphism* $\wedge : [1^3]\otimes([2^2]+[0]) \to \bigwedge^3[1^3]$ *is given by*

$$\begin{cases} 2[1^3]+[32]+[21]+[3^21]+[21^3]+[2^21]+[321^2]+[2^21^3] & (g\geqslant 5), \\ 2[1^3]+[32]+[21]+[3^21]+[21^3]+[2^21]+[321^2] & (g=4), \\ [1^3]+[32] & (g=3). \end{cases}$$

Proof. We can see that the highest weight vector $v_{[2^2]}$ of $[2^2]\subset \bigwedge^2 U_{\mathbb{Q}}$ is

$$\sum_{i=3}^{g} A_{12i}\wedge A_{12-i}$$

and the highest weight vector $v_{[0]}$ of $[0]\subset \bigwedge^2 U_{\mathbb{Q}}$ is

$$\sum_{1\leqslant i<j<k\leqslant g} A_{ijk}\wedge A_{-i-j-k} - \sum_{\substack{1\leqslant i<j\leqslant g \\ 1\leqslant k\leqslant g,\, k\neq i,j}} A_{ij-k}\wedge A_{-i-jk} + \sum_{\substack{1\leqslant i,j\leqslant g \\ i\neq j}} p_{ij}\wedge q_{ij}$$

by checking that they vanish when we apply $X_{i,j}$ $(i<j)$, $Y_{i,j}$ $(i<j)$ and U_i. Using $v_{[2^2]}$ and $v_{[0]}$, we construct some vectors in $\mathrm{Image}\wedge$ and decompose them to each irreducible component. To do so, we need a lot of $\mathfrak{sp}(2g,\mathbb{Q})$-vector spaces and $\mathfrak{sp}(2g,\mathbb{Q})$-homomorphisms. We summarize them in the following diagram:

$$
\begin{tikzcd}
\bigwedge^3[1^3]=\bigwedge^3 U_{\mathbb{Q}} \arrow[d, "\iota"'] & & \\
\bigwedge^3(\bigwedge^3 H_{\mathbb{Q}}) \arrow[r, "g_1"] \arrow[d, "f_1"'] & (\bigwedge^3 H_{\mathbb{Q}})\otimes(\bigwedge^2(\bigwedge^3 H_{\mathbb{Q}})) \arrow[d, "g_2"'] & \\
\bigotimes^3(\bigwedge^3 H_{\mathbb{Q}}) \arrow[d, "f_2"'] & (\bigwedge^3 H_{\mathbb{Q}})\otimes(\bigwedge^6 H_{\mathbb{Q}}) \arrow[d, "g_3"'] & \\
(\bigotimes^3 H_{\mathbb{Q}})\otimes(\bigwedge^3 H_{\mathbb{Q}})\otimes(\bigwedge^3 H_{\mathbb{Q}}) \arrow[d, "f_3"'] & (\bigwedge^3 H_{\mathbb{Q}})\otimes(\bigwedge^4 H_{\mathbb{Q}}) \arrow[r, "h_1"] \arrow[d, "g_4"'] & H_{\mathbb{Q}}\otimes(\bigwedge^2 H_{\mathbb{Q}})\otimes(\bigwedge^4 H_{\mathbb{Q}}) \arrow[d, "h_2"'] \\
H_{\mathbb{Q}}\otimes(\bigwedge^4 H_{\mathbb{Q}})\otimes(\bigwedge^4 H_{\mathbb{Q}}) \arrow[d, "f_4"'] & (\bigwedge^3 H_{\mathbb{Q}})\otimes(\bigwedge^2 H_{\mathbb{Q}}) \arrow[d, "g_5"'] & (\bigwedge^2 H_{\mathbb{Q}})\otimes(\bigwedge^5 H_{\mathbb{Q}}) \arrow[d, "h_3"'] \\
H_{\mathbb{Q}}\otimes(\bigwedge^2 H_{\mathbb{Q}})\otimes(\bigwedge^4 H_{\mathbb{Q}}) \arrow[d, "f_5"'] & H_{\mathbb{Q}}\otimes(\bigwedge^2 H_{\mathbb{Q}})\otimes(\bigwedge^2 H_{\mathbb{Q}}) \arrow[d, "g_6"'] & (\bigwedge^2 H_{\mathbb{Q}})\otimes(\bigwedge^3 H_{\mathbb{Q}}) \arrow[d, "h_4"'] \\
H_{\mathbb{Q}}\otimes(\bigwedge^2 H_{\mathbb{Q}})\otimes(\bigwedge^2 H_{\mathbb{Q}}) & H_{\mathbb{Q}}\otimes(\bigwedge^4 H_{\mathbb{Q}}) & (\bigwedge^2 H_{\mathbb{Q}})\otimes H_{\mathbb{Q}}
\end{tikzcd}
$$

where ι is the inclusion and homomorphisms f_i, g_i and h_i are defined as follows.

$$
\begin{cases}
f_1 = i^3_{\bigwedge^3 H_{\mathbb{Q}}},\\
f_2 = i^3_{H_{\mathbb{Q}}} \otimes 1 \otimes 1,\\
f_3(x_1\otimes x_2\otimes x_3\otimes A_{ijk}\otimes A_{lmn}) = x_1\otimes(x_2\wedge A_{ijk})\otimes(x_3\wedge A_{lmn}),\\
f_4 = 1\otimes C_4\otimes 1,\\
f_5 = 1\otimes 1\otimes C_2,
\end{cases}
$$

$$
\begin{cases}
g_1 = j_{\bigwedge^3 H_{\mathbb{Q}}},\\
g_2 = 1\otimes \psi^{2,3}_{H_{\mathbb{Q}}},\\
g_3 = 1\otimes C_6,\\
g_4 = 1\otimes C_4,\\
g_5 = j_{H_{\mathbb{Q}}}\otimes 1,\\
g_6 = 1\otimes \varphi^{2,2}_{H_{\mathbb{Q}}},
\end{cases}
$$

$$
\begin{cases}
h_1 = j_{H_{\mathbb{Q}}}\otimes 1,\\
h_2(x\otimes A_{ij}\otimes A_{klmn}) = A_{ij}\otimes(x\wedge A_{klmn}),\\
h_3 = 1\otimes C_5,\\
h_4 = 1\otimes C_3.
\end{cases}
$$

$[3^2 1]$: Taking the wedge product of $v_{[2^2]}$ and A_{123}, we obtain

$$
\sum_{i=3}^{g} A_{12i}\wedge A_{12-i}\wedge A_{123} \in \bigwedge^3 U_{\mathbb{Q}}.
$$

When $g \geqslant 4$, this is the non-zero highest weight vector of $[3^2 1]$. Hence we see that Image $\wedge$ contains the summand $[3^2 1]$.

$[321^2]$: Take the wedge product of $v_{[2^2]}$ and A_{134}. Then

$$\begin{aligned}
v_{[2^2]} \wedge A_{134} &= \sum_{i=3}^{g} A_{12i} \wedge A_{12-i} \wedge A_{134} \\
&\overset{f_1\circ\iota}{\longmapsto} \sum_{i=3}^{g} A_{12i} \otimes A_{12-i} \otimes A_{134} - \sum_{i=3}^{g} A_{12i} \otimes A_{134} \otimes A_{12-i} \\
&\qquad + \sum_{i=3}^{g} A_{12-i} \otimes A_{134} \otimes A_{12i} - \sum_{i=3}^{g} A_{12-i} \otimes A_{12i} \otimes A_{134} \\
&\qquad + \sum_{i=3}^{g} A_{134} \otimes A_{12i} \otimes A_{12-i} - \sum_{i=3}^{g} A_{134} \otimes A_{12-i} \otimes A_{12i} \\
&\overset{f_3\circ f_2}{\longmapsto} -\sum_{i=3}^{g} (a_1 \otimes A_{i12-i} \otimes A_{2134}) - \sum_{i=3}^{g} (a_1 \otimes A_{2134} \otimes A_{i12-i}) \\
&\qquad + \sum_{i=3}^{g} (a_1 \otimes A_{2134} \otimes A_{-i12i}) + \sum_{i=3}^{g} (a_1 \otimes A_{-i12i} \otimes A_{2134}) \\
&\qquad + \sum_{i=3}^{g} (a_1 \otimes A_{312i} \otimes A_{412-i} - a_1 \otimes A_{412i} \otimes A_{312-i}) \\
&\qquad - \sum_{i=3}^{g} (a_1 \otimes A_{312-i} \otimes A_{412i} - a_1 \otimes A_{412-i} \otimes A_{312i}) \\
&\overset{f_4}{\longmapsto} 2(g-1) a_1 \otimes A_{12} \otimes A_{1234}.
\end{aligned}$$

Hence we obtain

$$f_4 \circ f_3 \circ f_2 \circ f_1 \circ \iota (v_{[2^2]} \wedge A_{134}) = 2(g-1) a_1 \otimes A_{12} \otimes A_{1234}.$$

This is the highest weight vector of $[321^2]$ so that Image $\wedge$ contains the summand $[321^2]$ for $g \geqslant 4$.

$[32]$: Take the wedge product of $v_{[2^2]}$ and $p_{13} - p_{12} = A_{13-3} - A_{12-2}$. Then

$$\begin{aligned}
v_{[2^2]} \wedge (p_{13} - p_{12}) &= \sum_{i=3}^{g} A_{12i} \wedge A_{12-i} \wedge (p_{13} - p_{12}) \\
&\overset{f_1\circ\iota}{\longmapsto} \sum_{i=3}^{g} A_{12i} \otimes A_{12-i} \otimes (p_{13} - p_{12}) - \sum_{i=3}^{g} A_{12i} \otimes (p_{13} - p_{12}) \otimes A_{12-i} \\
&\qquad + \sum_{i=3}^{g} A_{12-i} \otimes (p_{13} - p_{12}) \otimes A_{12i} - \sum_{i=3}^{g} A_{12-i} \otimes A_{12i} \otimes (p_{13} - p_{12})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=3}^{g}(p_{13}-p_{12})\otimes A_{12i}\otimes A_{12-i}-\sum_{i=3}^{g}(p_{13}-p_{12})\otimes A_{12-i}\otimes A_{12i}\\
\xrightarrow{f_3\circ f_2}\ &-\sum_{i=3}^{g}a_1\otimes A_{i12-i}\otimes\bigl(a_2\wedge(p_{13}-p_{12})\bigr)\\
&-\sum_{i=3}^{g}a_1\otimes\bigl(a_2\wedge(p_{13}-p_{12})\bigr)\otimes A_{i12-i}\\
&+\sum_{i=3}^{g}a_1\otimes\bigl(a_2\wedge(p_{13}-p_{12})\bigr)\otimes A_{-i12i}\\
&+\sum_{i=3}^{g}a_1\otimes A_{-i12i}\otimes\bigl(a_2\wedge(p_{13}-p_{12})\bigr)\\
&+\sum_{i=3}^{g}a_1\otimes A_{312i}\otimes A_{-312-i}-\sum_{i=3}^{g}a_1\otimes A_{-312i}\otimes A_{312-i}\\
&-\sum_{i=3}^{g}a_1\otimes A_{312-i}\otimes A_{-312i}+\sum_{i=3}^{g}a_1\otimes A_{-312-i}\otimes A_{312i}\\
\xrightarrow{f_5\circ f_4}\ &-a_1\otimes(g-2)A_{12}\otimes(-A_{12})-a_1\otimes(-A_{12})\otimes(g-2)A_{12}\\
&+a_1\otimes(-A_{12})\otimes(2-g)A_{12}+a_1\otimes(2-g)A_{12}\otimes(-A_{12})\\
&-a_1\otimes(-A_{12})\otimes A_{12}-a_1\otimes A_{12}\otimes(-A_{12})\\
=\ &(4g-6)a_1\otimes A_{12}\otimes A_{12}.
\end{aligned}
$$

Hence we obtain

$$f_5\circ f_4\circ f_3\circ f_2\circ f_1\circ\iota\bigl(v_{[2^2]}\wedge(p_{13}-p_{12})\bigr)=(4g-6)a_1\otimes A_{12}\otimes A_{12}.$$

This is the highest weight vector of [32] so that Image$\wedge$ contains the summand [32] for $g\geqslant 3$.

$[2^2 1^3]$: Take the wedge product of $v_{[2^2]}$ and A_{345}. By a similar calculation, we obtain

$$h_2\circ h_1\circ g_3\circ g_2\circ g_1\circ\iota(v_{[2^2]}\wedge A_{345})=-3A_{12}\otimes A_{12345}.$$

This is the highest weight vector of $[2^2 1^3]$ so that Image$\wedge$ contains the summand $[2^2 1^3]$ for $g\geqslant 5$.

$[2^2 1]$: Take the wedge product of $v_{[2^2]}$ and $p_{34}-p_{32}=A_{34-4}-A_{32-2}$. Then we obtain

$$h_3\circ h_2\circ h_1\circ g_3\circ g_2\circ g_1\circ\iota\bigl(v_{[2^2]}\wedge(p_{34}-p_{32})\bigr)=(2g-9)A_{12}\otimes A_{123}.$$

This is the highest weight vector of $[2^2 1]$ so that Image$\wedge$ contains the summand $[2^2 1]$ for $g\geqslant 4$.

[21]: Take the wedge product of $v_{[2^2]}$ and $q_{21}-q_{23}=A_{-21-1}-A_{-23-3}$. Then we obtain

$$h_4\circ h_3\circ h_2\circ h_1\circ g_3\circ g_2\circ g_1\circ\iota\bigl(v_{[2^2]}\wedge(q_{21}-q_{23})\bigr)=-8(g-3)A_{12}\otimes a_1.$$

This is the highest weight vector of [21] so that Image $\wedge$ contains the summand [21] for $g \geqslant 4$.

$[21^3]$: Take the wedge product of $v_{[2^2]}$ and A_{-234}. Then we obtain

$$g_6 \circ g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(v_{[2^2]} \wedge A_{-234}) = 4a_1 \otimes A_{1234}.$$

This is the highest weight vector of $[21^3]$ so that Image $\wedge$ contains the summand $[21^3]$ for $g \geqslant 4$.

$[1^3]$: We now claim that Image $\wedge$ has the summand $[1^3]$ whose multiplicity is 2 for $g \geqslant 4$ and 1 for $g = 3$. To show it, consider following two homomorphisms

$$\bigwedge^3 U_{\mathbb{Q}} \xrightarrow{g_4\circ\dots\circ g_1\circ\iota} \Big(\bigwedge^3 H_{\mathbb{Q}}\Big) \otimes \Big(\bigwedge^2 H_{\mathbb{Q}}\Big) \xrightarrow{1\otimes C_2} \bigwedge^3 H_{\mathbb{Q}},$$
$$\bigwedge^3 U_{\mathbb{Q}} \xrightarrow{g_4\circ\dots\circ g_1\circ\iota} \Big(\bigwedge^3 H_{\mathbb{Q}}\Big) \otimes \Big(\bigwedge^2 H_{\mathbb{Q}}\Big) \xrightarrow{\varphi^{3,2}_{H_{\mathbb{Q}}}} \bigwedge^5 H_{\mathbb{Q}} \xrightarrow{C_5} \bigwedge^3 H_{\mathbb{Q}}.$$

We denote the former map by F and the latter one by G. By a similar calculation, we obtain

$$F(v_{[2^2]} \wedge A_{-1-23}) = 6A_{123},$$
$$G(v_{[2^2]} \wedge A_{-1-23}) = -4(g-3)A_{123}.$$

On the other hand, we prepare another vector $v_{[0]} \wedge A_{123}$. Then

$$\begin{aligned}
&v_{[0]} \wedge A_{123}\\
&\overset{g_1\circ\iota}{\longmapsto} \sum_{1\leqslant i<j<k\leqslant g} A_{ijk} \otimes A_{-i-j-k} \wedge A_{123} + \sum_{1\leqslant i<j<k\leqslant g} A_{-i-j-k} \otimes A_{123} \wedge A_{ijk}\\
&\qquad + \sum_{1\leqslant i<j<k\leqslant g} A_{123} \otimes A_{ijk} \wedge A_{-i-j-k}\\
&\qquad + \sum_{\substack{1\leqslant i<j\leqslant g\\ 1\leqslant k\leqslant g,\, k\neq i,j}} A_{ij-k} \otimes A_{-i-jk} \wedge A_{123}\\
&\qquad + \sum_{\substack{1\leqslant i<j\leqslant g\\ 1\leqslant k\leqslant g,\ k\neq i,j}} A_{-i-jk} \otimes A_{123} \wedge A_{ij-k}\\
&\qquad + \sum_{\substack{1\leqslant i<j\leqslant g\\ 1\leqslant k\leqslant g,\ k\neq i,j}} A_{123} \otimes A_{ij-k} \wedge A_{-i-jk}\\
&\qquad + \sum_{\substack{1\leqslant i,j\leqslant g\\ i\neq j}} (p_{ij} \otimes q_{ij} \wedge A_{123} + q_{ij} \otimes A_{123} \wedge p_{ij} + A_{123} \otimes p_{ij} \wedge q_{ij})\\
&\overset{g_2}{\longmapsto} \sum_{1\leqslant i<j<k\leqslant g} (A_{ijk} \otimes A_{-i-j-k123} + A_{-i-j-k} \otimes A_{123ijk} + A_{123} \otimes A_{ijk-i-j-k})\\
&\qquad + \sum_{\substack{1\leqslant i<j\leqslant g\\ 1\leqslant k\leqslant g,\ k\neq i,j}} (A_{ij-k} \otimes A_{-i-jk123}
\end{aligned}$$

$$
\begin{aligned}
&+ A_{-i-jk} \otimes A_{123ij-k} + A_{123} \otimes A_{ij-k-i-jk}) \\
&+ \sum_{\substack{1\leqslant i,j\leqslant g\\ i\neq j}} (p_{ij} \otimes q_{ij} \wedge A_{123} + q_{ij} \otimes A_{123} \wedge p_{ij} + A_{123} \otimes p_{ij} \wedge q_{ij}) \\
\overset{g_4\circ g_3}{\longmapsto}\ & A_{123} \otimes 2(A_{1-1} + A_{2-2} + A_{3-3}) \\
&+ \sum_{k=4}^{g} \bigl\{A_{13k} \otimes (-2A_{2-k}) + A_{12k} \otimes 2A_{3-k} + A_{23k} \otimes 2A_{1-k}\bigr\} \\
&+ \sum_{1\leqslant i<j<k\leqslant g} A_{123} \otimes \bigl\{-2(A_{i-i} + A_{j-j} + A_{k-k})\bigr\} \\
&- \sum_{k=4}^{g} \bigl\{A_{12-k} \otimes 2A_{3k} + A_{13-k} \otimes (-2A_{2k}) + A_{23-k} \otimes 2A_{1k}\bigr\} \\
&- \sum_{\substack{1\leqslant i<j\leqslant g\\ 1\leqslant k\leqslant g,\ k\neq i,j}} A_{123} \otimes 2(A_{i-i} + A_{j-j} + A_{k-k}) \\
&+ 2(p_{12} + p_{13}) \otimes A_{23} - 2(p_{21} + p_{23}) \otimes A_{13} + 2(p_{31} + p_{32}) \otimes A_{12} \\
&- A_{123} \otimes 6(g-2)\omega.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
F(v_{[0]} \wedge A_{123}) &= -2\bigl(2g^3 - 3g^2 - 2g - 3\bigr)A_{123}, \\
G(v_{[0]} \wedge A_{123}) &= \frac{-2(g-3)(2g^3 - 5g^2 + 7g + 2)}{g-1} A_{123}.
\end{aligned}
$$

Comparing images of $v_{[2^2]} \wedge A_{-1-23}$ and $v_{[0]} \wedge A_{123}$ under homomorphisms F and G, we can see that $v_{[2^2]} \wedge A_{-1-23}$ and $v_{[0]} \wedge A_{123}$ have $\mathfrak{sp}(2g, \mathbb{Q})$-linearly independent images in $[1^3]$ when $g \geqslant 4$. Therefore our claim follows.

From above computations, Proposition 6.2 follows. □

6.2. Summands which are not in the kernel

By considering the homomorphism

$$
\tau_* : H_3(\mathcal{I}_g, \mathbb{Q}) \to H_3(U, \mathbb{Q}) \cong \bigwedge^3 U_{\mathbb{Q}},
$$

we can obtain summands which are *not* in $\operatorname{Ker}\tau^*$ by decomposing the image of τ_* into each irreducible component as mentioned in the proof of Theorem 4.4. To obtain elements in $H_3(\mathcal{I}_g, \mathbb{Q})$, we construct some Abelian subgroups of $\mathcal{I}_g$ and take their fundamental classes.

We can find some Abelian groups by choosing simple closed curves on Σ_g along which Dehn twists are done.

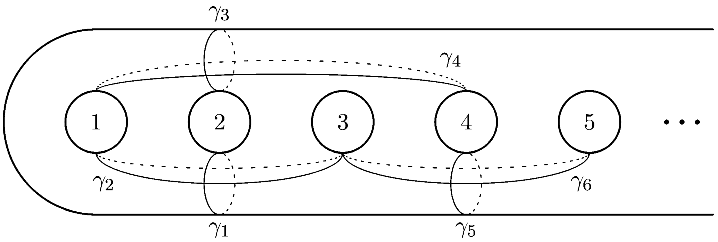


Fig. 4.

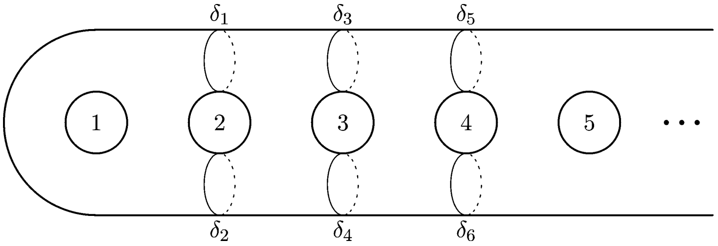


Fig. 5.

Lemma 6.3. *Following three elements*

$$w_1 = A_{123} \wedge A_{124} \wedge A_{345} \quad (g \geqslant 5),$$
$$w_2 = p_{21} \wedge \left(\sum_{i=1}^{2} p_{3i}\right) \wedge \left(\sum_{i=1}^{3} p_{4i}\right) \quad (g \geqslant 5),$$
$$w_3 = p_{12} \wedge p_{34} \wedge p_{56} \quad (g \geqslant 6)$$

are in $\mathrm{Image}(\tau_* : H_3(\mathcal{I}_g) \to H_3(U) \cong \bigwedge^3 U)$.

Proof. Construct homomorphisms $f_i : \mathbb{Z}^3 \to \mathcal{I}_g$ as follows, where $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, $x_3 = (0,0,1) \in \mathbb{Z}^3$ and T_c is the Dehn twist along a simple closed curve c.

- f_1: $f_1(x_1) = [T_{\gamma_1}, T_{\gamma_2}]$, $f_1(x_2) = [T_{\gamma_3}, T_{\gamma_4}]$, $f_1(x_3) = [T_{\gamma_5}, T_{\gamma_6}]$ where γ_i $(1 \leqslant i \leqslant 6)$ are simple closed curves as depicted in Fig. 4.
- f_2: $f_2(x_1) = T_{\delta_1} T_{\delta_2}^{-1}$, $f_2(x_2) = T_{\delta_3} T_{\delta_4}^{-1}$, $f_2(x_3) = T_{\delta_5} T_{\delta_6}^{-1}$ where δ_i $(1 \leqslant i \leqslant 6)$ are simple closed curves as depicted in Fig. 5.
- f_3: $f_3(x_1) = T_{\varepsilon_1} T_{\varepsilon_2}^{-1}$, $f_3(x_2) = T_{\varepsilon_3} T_{\varepsilon_4}^{-1}$, $f_3(x_3) = T_{\varepsilon_5} T_{\varepsilon_6}^{-1}$ where ε_i $(1 \leqslant i \leqslant 6)$ are simple closed curves as depicted in Fig. 6.

Since above curves are disjoint from each other, $f_i : \mathbb{Z}^3 \to \mathcal{I}_g$ are well-defined homomorphisms. By Lemma 2.1, we see that $(\tau \circ f_i)_* : H_3(\mathbb{Z}^3, \mathbb{Q}) \to H_3(U, \mathbb{Q}) = \bigwedge^3 U_{\mathbb{Q}}$ map the fundamental class of $H_3(\mathbb{Z}^3, \mathbb{Q})$ to homology classes $\widetilde{w_i} \in \bigwedge^3 U_{\mathbb{Q}}$ given by

$$\widetilde{w_1} = B_{123} \wedge B_{124} \wedge B_{345},$$

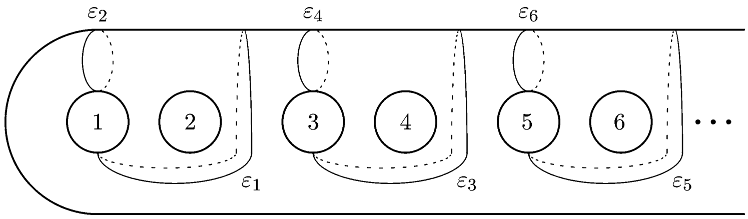


Fig. 6.

$$\widetilde{w_2} = q_{21} \wedge \left(\sum_{i=1}^{2} q_{3i}\right) \wedge \left(\sum_{i=1}^{3} q_{4i}\right),$$
$$\widetilde{w_3} = q_{12} \wedge q_{34} \wedge q_{56}.$$

Since τ is $\mathcal{M}_g$-equivariant, the lemma follows.

After the above preparation, we continue the proof of Theorem 5.2. We now assume $g \geqslant 5$. As in the previous subsection, we summarize $\mathfrak{sp}(2g, \mathbb{Q})$-vector spaces and $\mathfrak{sp}(2g, \mathbb{Q})$-homomorphisms in the following diagram:

$$\begin{tikzcd}
\bigwedge^3[1^3] = \bigwedge^3 U_{\mathbb{Q}} \arrow[d, "\iota"] & & \\
\bigwedge^3(\bigwedge^3 H_{\mathbb{Q}}) \arrow[r, "f_1"] \arrow[d, "g_1"] & \bigotimes^3(\bigwedge^3 H_{\mathbb{Q}}) \arrow[r, "f_3 \circ f_2"] \arrow[d, "f_6"] & H_{\mathbb{Q}} \otimes (\bigwedge^4 H_{\mathbb{Q}}) \otimes (\bigwedge^4 H_{\mathbb{Q}}) \\
(\bigwedge^3 H_{\mathbb{Q}}) \otimes (\bigwedge^2(\bigwedge^3 H_{\mathbb{Q}})) \arrow[d, "g_2"] & \bigotimes^2(H_{\mathbb{Q}} \otimes (\bigwedge^2 H_{\mathbb{Q}})) \otimes (\bigwedge^3 H_{\mathbb{Q}}) \arrow[r, "f_7"] & (\bigwedge^2 H_{\mathbb{Q}}) \otimes (\bigwedge^2 H_{\mathbb{Q}}) \otimes (\bigwedge^5 H_{\mathbb{Q}}) \\
(\bigwedge^3 H_{\mathbb{Q}}) \otimes (\bigwedge^6 H_{\mathbb{Q}}) \arrow[r, "g_7"] \arrow[d, "g_3"] & H_{\mathbb{Q}} \otimes (\bigwedge^2 H_{\mathbb{Q}}) \otimes (\bigwedge^6 H_{\mathbb{Q}}) \arrow[r, "g_8"] & (\bigwedge^2 H_{\mathbb{Q}}) \otimes (\bigwedge^7 H_{\mathbb{Q}}) \\
(\bigwedge^3 H_{\mathbb{Q}}) \otimes (\bigwedge^4 H_{\mathbb{Q}}) \arrow[r, "g_9"] \arrow[d, "g_4"] & H_{\mathbb{Q}} \otimes (\bigwedge^2 H_{\mathbb{Q}}) \otimes (\bigwedge^4 H_{\mathbb{Q}}) \arrow[r, "g_{10}"] \arrow[d, "g_{11}"] & (\bigwedge^2 H_{\mathbb{Q}}) \otimes (\bigwedge^5 H_{\mathbb{Q}}) \\
(\bigwedge^3 H_{\mathbb{Q}}) \otimes (\bigwedge^2 H_{\mathbb{Q}}) \arrow[d, "g_{12}"] & H_{\mathbb{Q}} \otimes (\bigwedge^6 H_{\mathbb{Q}}) & \\
(\bigwedge^3 H_{\mathbb{Q}}) \otimes H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \arrow[r, "g_{13}"] \arrow[d, "g_{14}"] & \bigwedge^5 H_{\mathbb{Q}} & \\
(\bigwedge^4 H_{\mathbb{Q}}) \otimes H_{\mathbb{Q}} & &
\end{tikzcd}$$

where new homomorphisms are given by

$$\begin{cases} f_6 = j_{H_{\mathbb{Q}}} \otimes j_{H_{\mathbb{Q}}} \otimes 1, \\ f_7(x_1 \otimes A_{kl} \otimes x_2 \otimes A_{mn} \otimes A_{pqr}) = A_{kl} \otimes A_{mn} \otimes (x_1 \wedge x_2 \wedge A_{pqr}), \end{cases}$$

$$\begin{cases} g_7 = j_{H_\mathbb{Q}} \otimes 1, \\ g_8(x \otimes A_{kl} \otimes A_{mnopqr}) = A_{kl} \otimes (x \wedge A_{mnopqr}), \\ g_9 = j_{H_\mathbb{Q}} \otimes 1, \\ g_{10}(x \otimes A_{kl} \otimes A_{pqrs}) = A_{kl} \otimes (x \wedge A_{pqrs}), \\ g_{11} = 1 \otimes \varphi^{2,4}_{H_\mathbb{Q}}, \\ g_{12} = 1 \otimes i^2_{H_\mathbb{Q}}, \\ g_{13}(A_{pqr} \otimes x_1 \otimes x_2) = A_{pqr} \wedge x_1 \wedge x_2, \\ g_{14} = \varphi^{3,1}_{H_\mathbb{Q}} \otimes 1. \end{cases}$$

$[32^3]$: From a direct calculation, we have

$$\begin{aligned} X_{1,5}w_1 &= A_{123} \wedge A_{124} \wedge A_{134} \\ &\xmapsto{f_1 \circ \iota} A_{123} \otimes A_{124} \otimes A_{134} - A_{123} \otimes A_{134} \otimes A_{124} + A_{124} \otimes A_{134} \otimes A_{123} \\ &\qquad - A_{124} \otimes A_{123} \otimes A_{134} + A_{134} \otimes A_{123} \otimes A_{124} - A_{134} \otimes A_{124} \otimes A_{123} \\ &\xmapsto{f_3 \circ f_2} 6a_1 \otimes A_{1234} \otimes A_{1234}. \end{aligned}$$

Hence we obtain

$$f_3 \circ f_2 \circ f_1 \circ \iota(X_{1,5}w_1) = 6A_{12} \otimes A_{1234} \otimes A_{1234}.$$

This is the highest weight vector of $[32^3]$ so that $[32^3]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 5$.

$[3^2 1^3]$: By a similar calculation, we have

$$f_7 \circ f_6 \circ f_1 \circ \iota(X_{2,4}X_{1,3}w_1) = 6A_{12} \otimes A_{12} \otimes A_{12345}.$$

This is the highest weight vector of $[3^2 1^3]$ so that $[3^2 1^3]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 5$.

As for $[1^9]$, $[1^7]$, $[1^3]$ and $[1^5]$, we calculate images of w_3 by following maps

$$\bigwedge^3 U_\mathbb{Q} \xrightarrow{\iota} \bigwedge^3 \Big(\bigwedge^3 H_\mathbb{Q}\Big) \xrightarrow{\psi^{3,3}_{H_\mathbb{Q}}} \bigwedge^9 H_\mathbb{Q} \xrightarrow{C_9} \bigwedge^7 H_\mathbb{Q} \xrightarrow{C_7} \bigwedge^5 H_\mathbb{Q} \xrightarrow{C_5} \bigwedge^3 H_\mathbb{Q}.$$

Then we obtain

$$\begin{aligned} w_3 &\xmapsto{\psi^{3,3}_{H_\mathbb{Q}} \circ \iota} p_{12} \wedge p_{34} \wedge p_{56} \\ &\xmapsto{C_9} 6(g-1)^{-1} A_{135} \wedge (A_{2-24-4} + A_{2-26-6} + A_{4-46-6}) \\ &\qquad - 10(g-1)^{-2} A_{135} \wedge (A_{2-2} + A_{4-4} + A_{6-6}) \wedge \omega \\ &\qquad + 12(g-1)^{-3} A_{135} \wedge \omega^2 \\ &\xmapsto{C_7} 2(g+19)(g-1)^{-2} A_{135} \wedge (A_{2-2} + A_{4-4} + A_{6-6}) \\ &\qquad - 6(g+11)(g-1)^{-3} A_{135} \wedge \omega \\ &\xmapsto{C_5} 12(5g+7)(g-1)^{-3} A_{135}. \end{aligned}$$

$[1^9]$: From the above calculation, we have

$$\psi^{3,3}_{H_\mathbb{Q}} \circ \iota(Y_{6,9}Y_{4,8}Y_{2,7}w_3) = A_{123456789}.$$

This is the highest weight vector of $[1^9]$ so that $[1^9]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 9$.

$[1^7]$: Similarly we have

$$C_9 \circ \psi_{H_{\mathbb{Q}}}^{3,3} \circ \iota(X_{6,8} Y_{4,7} Y_{2,8} w_3) = 6(g-1)^{-1} A_{1234567}.$$

This is the highest weight vector of $[1^7]$ so that $[1^7]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 8$.

$[1^3]$: We have

$$C_5 \circ C_7 \circ C_9 \circ \psi_{H_{\mathbb{Q}}}^{3,3} \circ \iota(X_{2,5} w_3) = -(60g+84)(g-1)^{-3} A_{123}.$$

This is the highest weight vector of $[1^3]$ so that $[1^3]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 3$.

Similar calculations can be applied to the other components except [1]. But we now omit the details.

$[2^3 1^3]$: We have

$$g_2 \circ g_1 \circ \iota(X_{1,6} Y_{1,6} Y_{2,6} Y_{3,5} w_2) = -A_{123} \otimes A_{123456}.$$

This is the highest weight vector of $[2^3 1^3]$ so that $[2^3 1^3]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 6$.

$[2^3 1]$: We have

$$g_3 \circ g_2 \circ g_1 \circ \iota(X_{2,4} X_{1,5} X_{4,5} Y_{1,5} Y_{3,5} w_2) = -3(g-1)^{-1}(g-3) A_{123} \otimes A_{1234}.$$

This is the highest weight vector of $[2^3 1]$ so that $[2^3 1]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 5$.

$[2^2 1]$: We have

$$g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{2,4} X_{1,5} X_{1,3} Y_{3,5} w_2) = 2(g-1)^{-1}(g-3)(g+1) A_{123} \otimes A_{12}.$$

This is the highest weight vector of $[2^2 1]$ so that $[2^2 1]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 5$.

$[2^2 1^5]$: We have

$$g_8 \circ g_7 \circ g_2 \circ g_1 \circ \iota(X_{1,3} Y_{1,7} Y_{2,6} Y_{3,5} w_2) = -A_{12} \otimes A_{1234567}.$$

This is the highest weight vector of $[2^2 1^5]$ so that $[2^2 1^5]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 7$.

$[2^2 1^3]$: We have

$$g_{10} \circ g_9 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{1,3} X_{1,6} Y_{2,6} Y_{3,5} w_2) = 3(g-1)^{-1} A_{12} \otimes A_{12345}.$$

This is the highest weight vector of $[2^2 1^3]$ so that $[2^2 1^3]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 6$.

$[21^5]$: We have

$$g_{11} \circ g_9 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{1,3} Y_{1,6} Y_{3,5} w_2) = 3(g-1)^{-1}(g-3) a_1 \otimes A_{123456}.$$

This is the highest weight vector of $[21^5]$ so that $[21^5]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 6$.

$[21^3]$: We have

$$\begin{aligned} &g_{14} \circ g_{12} \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{1,5} X_{1,3} Y_{3,5} w_2) \\ &\quad = 2(g-1)^{-2}(g-3)(g+1) A_{1234} \otimes a_1. \end{aligned}$$

This is the highest weight vector of $[21^3]$ so that $[21^3]$ is not in $\operatorname{Ker} \tau^*$ for $g \geqslant 5$.

$[1^5]$: We now claim that $\operatorname{Image} \tau_* \otimes \mathbb{Q}$ has $[1^5]$ whose multiplicity is actually 2 when $g \geqslant 7$ and 1 when $g = 5, 6$. We can see that

$$C_7 \circ C_9 \circ \psi_{H_{\mathbb{Q}}}^{3,3} \circ \iota(Y_{1,5} w_2) = -22(g-1)^{-2}(g-5)(g-6) A_{12345},$$

$$g_{13} \circ g_{12} \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(Y_{1,5} w_2) = 4(g-1)^{-2}(g+1)(4g-15) A_{12345}.$$

On the other hand, we prepare another vector $X_{4,7}Y_{2,7}w_3$ which is defined when $g \geqslant 7$. Then

$$C_7 \circ C_9 \circ \psi^{3,3}_{H_{\mathbb{Q}}} \circ \iota(X_{4,7}Y_{2,7}w_3) = -2(g-1)^{-2}(g+19)A_{12345},$$

$$g_{13} \circ g_{12} \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{4,7}Y_{2,7}w_3) = -4(g-1)^{-2}(g+1)A_{12345}.$$

From these, we can check that $Y_{1,5}w_2$ and $X_{4,7}Y_{2,7}w_3$ have $\mathfrak{sp}(2g,\mathbb{Q})$-linearly independent images in $[1^5]$ when $g \geqslant 7$. Therefore our claim follows.

From above computations, we have determined the image of τ^* except the summand [1]. $\square$

6.3. *The summand* [1] *and characteristic classes of surface bundles on the pointed Torelli group*

In this subsection, we treat the summand [1]. As a result, we see that this summand embodies one of characteristic classes of surface bundles.

Let $v_{[1]}$ be the highest weight vector of [1] in $\bigwedge^3 U_{\mathbb{Q}}$. Note that $v_{[1]}$ is in $\operatorname{Ker}\tau^*$ if and only if [1] is contained in $\operatorname{Ker}\tau^*$. By the universal coefficient theorem, $v_{[1]}$ can be considered as an element of $\operatorname{Hom}(H_3(U),\mathbb{Q})$. Then this homomorphism factors through $H_{\mathbb{Q}}$ as follows:

$$v_{[1]} : H_3 U \cong \bigwedge^3 U \longrightarrow \bigwedge^3 U_{\mathbb{Q}} \xrightarrow{\ p\ } H_{\mathbb{Q}} \xrightarrow{\ b_1^*\ } \mathbb{Q}$$

where the first map is an injection and p is an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant projection $\bigwedge^3 U_{\mathbb{Q}} \to [1] = H_{\mathbb{Q}}$. Then

$$\tau^*(v_{[1]}) = 0 \in H^3(\mathcal{I}_g,\mathbb{Q}) \iff \tau^*(p) = p \circ \tau_* = 0 \in \operatorname{Hom}\bigl(H_3(\mathcal{I}_g), H_{\mathbb{Q}}\bigr).$$

Since $\mathcal{I}_g$ acts on $H_{\mathbb{Q}}$ trivially, we see that $\tau^*(p) \in \operatorname{Hom}(H_3(\mathcal{I}_g), H_{\mathbb{Q}}) \cong H^3(\mathcal{I}_g, H_{\mathbb{Q}})$. The following result admits us to consider $\tau^*(p)$ to be an element of $H^4(\mathcal{I}_{g,*},\mathbb{Q})$.

Theorem 6.4 (Kawazumi, Morita [9]). *The cohomology group $H^*(\mathcal{I}_{g,*},\mathbb{Q})$ has a canonical decomposition of*

$$H^*(\mathcal{I}_{g,*},\mathbb{Q}) \cong H^*(\mathcal{I}_g,\mathbb{Q}) \oplus H^{*-1}(\mathcal{I}_g, H_{\mathbb{Q}}) \oplus H^{*-2}(\mathcal{I}_g,\mathbb{Q}).$$

Explicitly, the inclusion $j : H^{*-1}(\mathcal{I}_g, H_{\mathbb{Q}}) \hookrightarrow H^*(\mathcal{I}_{g,*},\mathbb{Q})$ is the composite of

$$\begin{aligned} H^{*-1}(\mathcal{I}_g, H_{\mathbb{Q}}) &\longrightarrow H^{*-1}(\mathcal{I}_{g,*}, H_{\mathbb{Q}}) \xrightarrow{\cup\chi} H^*(\mathcal{I}_{g,*}, H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}) \\ &\xrightarrow{\mu_*} H^*(\mathcal{I}_{g,*},\mathbb{Q}) \end{aligned}$$

where $\chi \in H^1(\mathcal{I}_{g,*}, H_{\mathbb{Q}}) \cong \operatorname{Hom}(H_1(\mathcal{I}_{g,*}), H_{\mathbb{Q}})$ is the $\mathcal{M}_{g,*}$-equivariant homomorphism given by the composition of the Johnson homomorphism $\tau : H_1(\mathcal{I}_{g,*}) \to \bigwedge^3 H_{\mathbb{Q}}$ and the contraction $C_3 : \bigwedge^3 H_{\mathbb{Q}} \to H_{\mathbb{Q}}$, and the last map is applying the intersection form μ to coefficients in $H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}$.

We refer to Theorem 5.5 in [9] for the proof of this theorem. From this result, we show

Lemma 6.5. *The following diagram commutes*:

$$\begin{tikzcd}
\mathrm{Hom}(\bigwedge^3 U_{\mathbb{Q}}, H_{\mathbb{Q}}) \arrow[r, "\tau^*"] \arrow[d, equal] & \mathrm{Hom}(H_3(\mathcal{I}_g), H_{\mathbb{Q}}) \arrow[d, equal] \\
(\bigwedge^3 U_{\mathbb{Q}}) \otimes H_{\mathbb{Q}} \arrow[d, "i"'] & H^3(\mathcal{I}_g, H_{\mathbb{Q}}) \arrow[d, "j"'] \\
\bigwedge^4(\bigwedge^3 H_{\mathbb{Q}}) \arrow[r, "\tau^*"] & H^4(\mathcal{I}_{g,*}, \mathbb{Q})
\end{tikzcd}$$

where the left vertical map $i : (\bigwedge^3 U_{\mathbb{Q}}) \otimes H_{\mathbb{Q}} \to \bigwedge^4(\bigwedge^3 H_{\mathbb{Q}})$ *is the canonical inclusion with respect to the decomposition*

$$\bigwedge^4\Big(\bigwedge^3 H_{\mathbb{Q}}\Big) = \Big(\bigwedge^4 U_{\mathbb{Q}}\Big) \oplus \Big(\Big(\bigwedge^3 U_{\mathbb{Q}}\Big) \otimes H_{\mathbb{Q}}\Big) \oplus \Big(\Big(\bigwedge^2 U_{\mathbb{Q}}\Big) \otimes \Big(\bigwedge^2 H_{\mathbb{Q}}\Big)\Big)$$
$$\oplus \Big(U_{\mathbb{Q}} \otimes \Big(\bigwedge^3 H_{\mathbb{Q}}\Big)\Big) \oplus \Big(\bigwedge^4 H_{\mathbb{Q}}\Big).$$

Proof. First, we check that the following diagram

$$\begin{tikzcd}
H^3(U_{\mathbb{Q}}, H_{\mathbb{Q}}) \arrow[r, "\tau^*"] \arrow[d] & H^3(\mathcal{I}_g, H_{\mathbb{Q}}) \arrow[d] \\
H^3(\bigwedge^3 H_{\mathbb{Q}}, H_{\mathbb{Q}}) \arrow[r, "\tau^*"] \arrow[d, "\cup \omega_0"] & H^3(\mathcal{I}_{g,*}, H_{\mathbb{Q}}) \arrow[d, "\cup \chi"] \\
H^4(\bigwedge^3 H_{\mathbb{Q}}, H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}) \arrow[r, "\tau^*"] \arrow[d, "\mu_*"] & H^4(\mathcal{I}_{g,*}, H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}) \arrow[d, "\mu_*"] \\
H^4(\bigwedge^3 H_{\mathbb{Q}}, \mathbb{Q}) \arrow[r, "\tau^*"] & H^4(\mathcal{I}_{g,*}, \mathbb{Q})
\end{tikzcd}$$

commutes, where $\omega_0 \in H^1(\bigwedge^3 H_{\mathbb{Q}}, H_{\mathbb{Q}}) = \mathrm{Hom}(\bigwedge^3 H_{\mathbb{Q}}, H_{\mathbb{Q}})$ is the contraction C_3. Indeed, the commutativity of top and bottom squares follows from properties of cohomology of groups. As for the middle one, its commutativity follows from definitions of ω_0 and χ. Note that the composite of right vertical maps is j by definition. On the other hand, since ω_0 is $\mathrm{Sp}(2g, \mathbb{Q})$-equivariant homomorphism, we see that

$$\omega_0 = \sum_{i=1}^{g} \big\{(a_i \wedge \omega) \otimes b_i - (b_i \wedge \omega) \otimes a_i\big\}$$
$$\in \Big(\bigwedge^3 H_{\mathbb{Q}}\Big) \otimes H_{\mathbb{Q}} \cong \mathrm{Hom}\Big(\bigwedge^3 H_{\mathbb{Q}}, H_{\mathbb{Q}}\Big).$$

Then left vertical maps can be rewritten as follows:

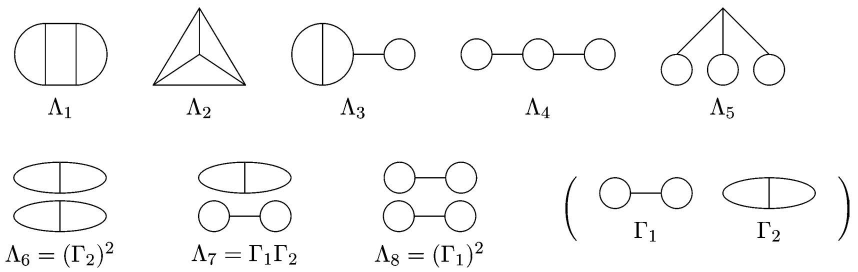


Fig. 7.

$$\Big(\bigwedge^3 U_{\mathbb{Q}}\Big) \otimes H_{\mathbb{Q}} \longrightarrow \Big(\bigwedge^3\Big(\bigwedge^3 H_{\mathbb{Q}}\Big)\Big) \otimes H_{\mathbb{Q}}$$
$$\xrightarrow{\otimes\omega_0} \Big(\bigwedge^3\Big(\bigwedge^3 H_{\mathbb{Q}}\Big)\Big) \otimes H_{\mathbb{Q}} \otimes \Big(\bigwedge^3 H_{\mathbb{Q}}\Big) \otimes H_{\mathbb{Q}} \longrightarrow \bigwedge^4\Big(\bigwedge^3 H_{\mathbb{Q}}\Big)$$

where the first map is the injection and the third map is applying the contraction C_3 to the second and fourth components of $(\bigwedge^3(\bigwedge^3 H_{\mathbb{Q}})) \otimes H_{\mathbb{Q}} \otimes (\bigwedge^3 H_{\mathbb{Q}}) \otimes H_{\mathbb{Q}}$. From this, we can see that the composite of these maps coincides with the injection i. Therefore the lemma follows.

The homomorphism $p : \bigwedge^3 U_{\mathbb{Q}} \to H_{\mathbb{Q}}$ is $\mathrm{Sp}(2g, \mathbb{Q})$-equivariant so that it can be considered to be an $\mathrm{Sp}(2g, \mathbb{Q})$-invariant vector as an element of the $\mathrm{Sp}(2g, \mathbb{Q})$-vector space $\mathrm{Hom}(\bigwedge^3 U_{\mathbb{Q}}, H_{\mathbb{Q}})$. Now we use the following commutative diagram

$$\begin{tikzcd}
\mathbb{Q}[\Gamma;\, \Gamma \in \mathcal{G}]^{(4)} \arrow[r, "\Phi_\alpha"] & (\bigwedge^4(\bigwedge^3 H_{\mathbb{Q}}))^{\mathrm{Sp}} \arrow[r, "\tilde{k}^*"] \arrow[d] & H^4(\mathcal{M}_{g,*}, \mathbb{Q}) \arrow[d] \\
 & \bigwedge^4(\bigwedge^3 H_{\mathbb{Q}}) \arrow[r, "\tau^*"] & H^4(\mathcal{I}_{g,*}, \mathbb{Q})
\end{tikzcd}$$

constructed by Morita in [14] (see also Sections 2 and 3 in [9]). In the diagram, $\tilde{k}^*$ is a homomorphism induced from the extended Johnson homomorphism ρ_1 and $\Phi_\alpha : \mathbb{Q}[\Gamma;\, \Gamma \in \mathcal{G}]^{(4)} \to (\bigwedge^4(\bigwedge^3 H_{\mathbb{Q}}))^{\mathrm{Sp}}$ is the degree 4 part of the algebra homomorphism

$$\Phi_\alpha : \mathbb{Q}[\Gamma;\, \Gamma \in \mathcal{G}] \to \Big(\bigwedge^*\Big(\bigwedge^3 H_{\mathbb{Q}}\Big)\Big)^{\mathrm{Sp}}$$

from the polynomial algebra $\mathbb{Q}[\Gamma;\, \Gamma \in \mathcal{G}]$ generated by the set $\mathcal{G}$ of isomorphism classes of connected trivalent graphs to $\mathrm{Sp}(2g, \mathbb{Q})$-invariant part of $\bigwedge^*(\bigwedge^3 H_{\mathbb{Q}})$. The degree of a connected trivalent graph is given by its cardinality of vertices. Explicitly, the degree 4 part $\mathbb{Q}[\Gamma;\, \Gamma \in \mathcal{G}]^{(4)}$ is a $\mathbb{Q}$-vector space with basis $\langle \Lambda_i;\ 1 \leqslant i \leqslant 8\rangle$ where Λ_i are trivalent graphs as in Fig. 7 (as for Λ_6, Λ_7 and Λ_8, they are products of two trivalent graphs of degree 2). Then we see that $p \in \bigwedge^4(\bigwedge^3 H_{\mathbb{Q}})$ comes from an element of the invariant part

$$\Big(\Big(\bigwedge^3 U_{\mathbb{Q}}\Big) \otimes H_{\mathbb{Q}}\Big)^{\mathrm{Sp}} \subset \Big(\bigwedge^4\Big(\bigwedge^3 H_{\mathbb{Q}}\Big)\Big)^{\mathrm{Sp}}$$

which is 1-dimensional for $g \geqslant 5$. The top horizontal map $\tilde{k}^* \circ \Phi_\alpha$ in the above diagram is completely determined by Kawazumi and Morita in [9] where they give $\alpha_\Lambda := \tilde{k}^* \circ \Phi_\alpha(\Lambda)$

for trivalent graphs Λ explicitly. Hence we need to write the unique invariant vector of $((\bigwedge^3 U_{\mathbb{Q}}) \otimes H_{\mathbb{Q}})^{\mathrm{Sp}}$ by using them.

Lemma 6.6. *Up to non-zero scalars, the equality*

$$\begin{aligned} p = {} & 4(g-1)^3 a_{\Lambda_3} + 12(g-1)^2 a_{\Lambda_4} + 2(g+5)(g-1) a_{\Lambda_5} + 2(g-1)^2 a_{\Lambda_7} \\ & + (7g-1) a_{\Lambda_8} \end{aligned}$$

holds where we write a_Λ for the $\mathrm{Sp}(2g,\mathbb{Q})$*-invariant tensor which associates to the trivalent graph Λ; namely $a_\Lambda = \Phi_\alpha(\Lambda)$.*

Proof. Since $((\bigwedge^3 U_{\mathbb{Q}}) \otimes H_{\mathbb{Q}})^{\mathrm{Sp}} \subset (\bigwedge^4(\bigwedge^3 H_{\mathbb{Q}}))^{\mathrm{Sp}}$ is 1-dimensional for $g \geqslant 5$, we obtain the result by decomposing some elements of $(\bigwedge^4(\bigwedge^3 H_{\mathbb{Q}}))^{\mathrm{Sp}}$ with respect to the direct sum decomposition

$$\begin{aligned} \Big(\bigwedge^4\Big(\bigwedge^3 H_{\mathbb{Q}}\Big)\Big)^{\mathrm{Sp}} = {} & \Big(\bigwedge^4 U_{\mathbb{Q}}\Big)^{\mathrm{Sp}} \oplus \Big(\Big(\bigwedge^3 U_{\mathbb{Q}}\Big) \otimes H_{\mathbb{Q}}\Big)^{\mathrm{Sp}} \oplus \Big(\Big(\bigwedge^2 U_{\mathbb{Q}}\Big) \\ & \otimes \Big(\bigwedge^2 H_{\mathbb{Q}}\Big)\Big)^{\mathrm{Sp}} \oplus \Big(U_{\mathbb{Q}} \otimes \Big(\bigwedge^3 H_{\mathbb{Q}}\Big)\Big)^{\mathrm{Sp}} \oplus \Big(\bigwedge^4 H_{\mathbb{Q}}\Big)^{\mathrm{Sp}} \end{aligned}$$

and seeing the component of $((\bigwedge^3 U_{\mathbb{Q}}) \otimes H_{\mathbb{Q}})^{\mathrm{Sp}}$. We refer to Section 3 in [17] for a method for obtaining such a decomposition. We now decompose Λ_2. In the rest of the proof, we use the notation of [17], while we omit their definitions. All we have to do is to calculate $a^{3,1}_{\Lambda_2} \in ((\bigwedge^3 U_{\mathbb{Q}}) \otimes H_{\mathbb{Q}})^{\mathrm{Sp}}$ from a_{Λ_2}. By Lemma 3.2 in [17], we see that

$$a^{3,1}_{\Lambda_2} = a^{(1)}_{\Lambda_2} - 2\,a^{(2)}_{\Lambda_2} + 3\,a^{(3)}_{\Lambda_2} - 4\,a^{(4)}_{\Lambda_2}$$

where $a^{(i)}_{\Lambda_2}$ are obtained from the formula in Proposition 3.6 in [17]. A somewhat long calculation shows that

$$\begin{aligned} a^{(1)}_{\Lambda_2} &= \frac{-12}{2g-2} a_{\Lambda_3}, \\ a^{(2)}_{\Lambda_2} &= \frac{6}{(2g-2)^2}(6a_{\Lambda_4} + 2a_{\Lambda_5} + a_{\Lambda_7}), \\ a^{(3)}_{\Lambda_2} &= \frac{-4}{(2g-2)^3}\big\{(14-2g)a_{\Lambda_5} + 12a_{\Lambda_8}\big\}, \\ a^{(4)}_{\Lambda_2} &= \frac{66-30g}{(2g-2)^4} a_{\Lambda_8}. \end{aligned}$$

From this, we obtain

$$\begin{aligned} a^{3,1}_{\Lambda_2} = {} & -\frac{6}{g-1}\,a_{\Lambda_3} - \frac{18}{(g-1)^2}\,a_{\Lambda_4} - \frac{3(g+5)}{(g-1)^3}\,a_{\Lambda_5} \\ & - \frac{3}{(g-1)^2}\,a_{\Lambda_7} - \frac{3(7g-1)}{2(g-1)^4}\,a_{\Lambda_8} \end{aligned}$$

and the lemma follows. $\square$

By results of Kawazumi and Morita, we have

$$\begin{cases} \alpha_{\Gamma_1} = -e_1 - 4g(g-1)e, \\ \alpha_{\Gamma_2} = -e_1 + 6ge, \end{cases}$$

and

$$\begin{cases} \alpha_{\Lambda_1} = e_2 + 6ee_1 + \bigl(4g^2 - 20g - 2\bigr)e^2, \\ \alpha_{\Lambda_2} = e_2 + 6ee_1 - (22g+2)e^2, \\ \alpha_{\Lambda_3} = e_2 - (2g-5)ee_1 + \bigl(16g^2 - 14g - 2\bigr)e^2, \\ \alpha_{\Lambda_4} = e_2 - (4g-4)ee_1 - \bigl(8g^3 - 20g^2 + 10g + 2\bigr)e^2, \\ \alpha_{\Lambda_5} = e_2 - (6g-3)ee_1 - \bigl(16g^3 - 24g^2 + 6g + 2\bigr)e^2, \\ \alpha_{\Lambda_6} = (\alpha_{\Gamma_2})^2 = e_1^2 - 12ee_1 + 36g^2e^2, \\ \alpha_{\Lambda_7} = \alpha_{\Gamma_1} \cdot \alpha_{\Gamma_2} = e_1^2 + \bigl(4g^2 - 10g\bigr)ee_1 - 24g^2(g-1)e^2, \\ \alpha_{\Lambda_8} = (\alpha_{\Gamma_1})^2 = e_1^2 + 8g(g-1)ee_1 + 16g^2(g-1)^2e^2, \end{cases}$$

which are obtained from Example 1.4 and Table 8.1 in [9]. From this,

$$\begin{aligned} \rho_1^*(p) = {} & 2(g-1)(2g+1)(g+1)e_2 + 4(g-1)^2(2g+1)(g+1)e^2 \\ & + (2g+1)(g+1)e_1^2 + (g-1)(2g+1)(g+1)ee_1. \end{aligned}$$

Since e_1 vanishes on $H^*(\mathcal{I}_{g,*}, \mathbb{Q})$, we obtain

$$\tau^*(p) = 2(g-1)(2g+1)(g+1)\bigl\{e_2 - (2-2g)e^2\bigr\}.$$

Therefore

$$\tau^*(v_{[1]}) = 0 \in H^3(\mathcal{I}_g, \mathbb{Q}) \iff e_2 - (2-2g)e^2 = 0 \in H^4(\mathcal{I}_{g,*}, \mathbb{Q}).$$

This completes the proof of Theorem 5.3. □

Finally, we make some remarks. The same argument as above is valid for summands [1] of higher degrees in $\bigwedge^* U_{\mathbb{Q}}$. Hence we see that summands [1] in $\bigwedge^* U_{\mathbb{Q}}$ represent some relations of the Euler class and Morita–Mumford classes on the Torelli group, although we do not know whether they are trivial or not at the present. As for the degree 3, which we treated above, using the well-known fact about the realization of homology classes, we see that the problem is reduced to the case of three-dimensional manifold groups, which are fundamental groups of some three-dimensional closed manifolds. Thus we have given a new approach to the non-triviality problem of Morita–Mumford classes on the Torelli group.

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